

TATE-POITOU DUALITY AND THE FIBER OF THE CYCLOTOMIC TRACE FOR THE SPHERE SPECTRUM

ANDREW J. BLUMBERG AND MICHAEL A. MANDELL

ABSTRACT. Let $p \in \mathbb{Z}$ be an odd prime. We identify the homotopy type of the homotopy fiber of the cyclotomic trace for the sphere spectrum in terms of the algebraic K -theory of \mathbb{Z} . To do this, we prove a spectral version of Tate-Poitou duality for the algebraic K -theory spectra of number rings with p inverted. This identifies the homotopy type of the fiber of the cyclotomic trace $K(\mathcal{O}_F)_p^\wedge \rightarrow TC(\mathcal{O}_F)_p^\wedge$ after taking a suitably connective cover. As a further application, we prove a conjecture of Calegari about the completed cohomology of congruence subgroups.

INTRODUCTION

The h -cobordism theorem of Smale states that for a simply-connected smooth manifold M of dimension ≥ 5 , every h -cobordism is diffeomorphic to the trivial h -cobordism $M \times [0, 1]$. When the simple-connectivity condition is relaxed, there is an obstruction to an h -cobordism being trivial; the s -cobordism theorem of Barden, Mazur, and Stallings identifies the obstruction in terms of Whitehead torsion, which lives in a quotient of the algebraic K -group $K_1(\mathbb{Z}[\pi_1 M])$. Hatcher took the perspective that the s -cobordism theorem should be regarded as the isomorphism on components induced from an algebraic description of a *space* of h -cobordisms; this was the subject of his stable parametrized h -cobordism theorem. Work of Hatcher and Igusa established comparisons between stable pseudo-isotopy theory, the stable concordance space, and a topologized space of h -cobordisms. Waldhausen outlined a different proof of the stable parametrized h -cobordism theorem (now fleshed out with Jahren and Rognes in [44]) that relates the space of h -cobordisms to algebraic K -theory.

Specifically, Waldhausen constructed a homotopy functor called “the algebraic K -theory of the space M ” as the algebraic K -theory of a ring spectrum enhancing the group ring $\mathbb{Z}[\pi_1 M]$. Regarding the based loop space ΩM as a generalization of a group and the sphere spectrum \mathbb{S} as an enhanced form of ring, one can form the spherical group ring $\mathbb{S}[\Omega M] = \Sigma_+^\infty \Omega M$. The spherical group ring has the property that π_0 is the group ring $\mathbb{Z}[\pi_1 M]$, but it is more closely related to the homotopy type of M when M is not aspherical (and has the same relationship to $\mathbb{Z}[\pi_1 M]$

Date: October 24, 2016.

2010 Mathematics Subject Classification. Primary 19D10, 19F05.

Key words and phrases. Algebraic K -theory of spaces, algebraic K -theory of number rings, cyclotomic trace, Tate-Poitou duality, Artin-Verdier duality, descent spectral sequence, Whitehead tower.

The first author was supported in part by NSF grant DMS-1151577.

The second author was supported in part by NSF grant DMS-1505579.

as \mathbb{S} has to \mathbb{Z} when M is aspherical). Waldhausen's work identifies the stable h -cobordism space as the underlying infinite loop space of the fiber of an assembly map $K(\mathbb{S}) \wedge M_+ \rightarrow K(\mathbb{S}[M])$ for M a PL-manifold. When M is a smooth manifold, the composite map $\mathbb{S} \wedge M_+ \rightarrow K(\mathbb{S}[M])$ splits and the underlying infinite loop space of the complementary summand is the smooth Whitehead space; its double loop space is the space of stable concordances.

The most basic example of the algebraic K -theory of a manifold arises when we take $M = D^n$. In this case, the spherical group ring is just the sphere spectrum \mathbb{S} and Waldhausen's algebraic K -theory is $K(\mathbb{S})$, the algebraic K -theory of the sphere spectrum. The problem of providing a description of the homotopy type of $K(\mathbb{S})$ has motivated lots of interesting work in algebraic K -theory over the past thirty years.

Our understanding of algebraic K -theory of ring spectra has been shaped by the introduction of trace methods. Bökstedt, Hsiang, and Madsen constructed a topological version of negative cyclic homology, topological cyclic homology (TC), and a Chern character $K \rightarrow TC$, the cyclotomic trace. Topological cyclic homology is approachable using the methods of equivariant stable homotopy theory; for example, trace methods were used to great effect by Hesselholt and Madsen to calculate the algebraic K -theory of rings in characteristic p and of local fields and their integers [25, 26, 27] and by Rognes to understand the homotopy type of $K(\mathbb{S})$ at regular primes [38, 39].

The lesson of Rognes' work is that the central question involved in understanding the homotopy type of $K(\mathbb{S})$ is understanding the homotopy fiber of the p -completed cyclotomic trace $\tau: K(\mathbb{S})_p^\wedge \rightarrow TC(\mathbb{S})_p^\wedge$; we denote this p -completed homotopy fiber as $\text{Fib}(\tau)$. Building on work of Goodwillie and McCarthy, a basic theorem of Dundas provides a homotopy cartesian square

$$\begin{array}{ccc} K(\mathbb{S}) & \longrightarrow & K(\mathbb{Z}) \\ \downarrow & & \downarrow \\ TC(\mathbb{S}) & \longrightarrow & TC(\mathbb{Z}), \end{array}$$

where the maps $K(\mathbb{S}) \rightarrow K(\mathbb{Z})$ and $TC(\mathbb{S}) \rightarrow TC(\mathbb{Z})$ are the linearization maps induced by the zeroth Postnikov section. This identifies $\text{Fib}(\tau)$ also as the homotopy fiber of cyclotomic trace $K(\mathbb{Z})_p^\wedge \rightarrow TC(\mathbb{Z})_p^\wedge$, which (as we explain below) we can study using arithmetic methods. The result is that we can describe $\text{Fib}(\tau)$ in terms of Brown-Comenetz (or Anderson) duality and $K(1)$ -localization (where $K(1)$ denotes the first Morava K -theory spectrum at the prime p). The \mathbb{Z}_p -Anderson dual $I_{\mathbb{Z}_p}X$ of a p -complete spectrum X may be constructed as the derived function spectrum $F(X, I_{\mathbb{Z}_p}\mathbb{S})$ where $I_{\mathbb{Z}_p}\mathbb{S}$ is the p -completion of the desuspension of the Brown-Comenetz dual of the sphere spectrum

$$I_{\mathbb{Z}_p}\mathbb{S} = (\Sigma^{-1}I_{\mathbb{Q}/\mathbb{Z}}\mathbb{S})_p^\wedge.$$

The Brown-Comenetz dual of the sphere spectrum $I_{\mathbb{Q}/\mathbb{Z}}\mathbb{S}$ represents the Pontryagin dual of π_0 :

$$[X, I_{\mathbb{Q}/\mathbb{Z}}\mathbb{S}] \cong \text{Hom}(\pi_0 X, \mathbb{Q}/\mathbb{Z})$$

for any spectrum X (where $[-, -]$ denotes maps in the stable category). We identify $\text{Fib}(\tau)$ as follows.

Main Theorem. *Let p be an odd prime. The connective cover of the homotopy fiber $\text{Fib}(\tau)$ of the cyclotomic trace $K(\mathbb{S})_p^\wedge \rightarrow TC(\mathbb{S})_p^\wedge$ is canonically weakly equivalent to the connective cover of $\Sigma^{-1}I_{\mathbb{Z}_p}(L_{K(1)}K(\mathbb{Z}))$.*

The weak equivalence of the previous theorem is the connective cover of a weak equivalence $L_{K(1)}\text{Fib}(\tau) \rightarrow \Sigma^{-1}I_{\mathbb{Z}_p}(L_{K(1)}K(\mathbb{Z}))$ constructed as follows. The map $K(\mathbb{Z})_p^\wedge \rightarrow TC(\mathbb{Z})_p^\wedge$ is a map of commutative \mathbb{S} -algebras (E_∞ ring spectra) and so the homotopy fiber $\text{Fib}(\tau)$ has a canonical $K(\mathbb{Z})_p^\wedge$ -module structure that induces a $L_{K(1)}K(\mathbb{Z})$ -module structure on $L_{K(1)}\text{Fib}(\tau)$. The composite of the multiplication and a canonical map in the stable category $L_{K(1)}\text{Fib}(\tau) \rightarrow \Sigma^{-1}I_{\mathbb{Z}_p}\mathbb{S}$ induce a pairing

$$L_{K(1)}K(\mathbb{Z}) \wedge L_{K(1)}\text{Fib}(\tau) \longrightarrow L_{K(1)}\text{Fib}(\tau) \longrightarrow \Sigma^{-1}I_{\mathbb{Z}_p}\mathbb{S}$$

that then induces the map

$$L_{K(1)}\text{Fib}(\tau) \longrightarrow F(L_{K(1)}K(\mathbb{Z}), \Sigma^{-1}I_{\mathbb{Z}_p}\mathbb{S}) = \Sigma^{-1}I_{\mathbb{Z}_p}K(\mathbb{Z}).$$

We discuss the map $L_{K(1)}\text{Fib}(\tau) \rightarrow \Sigma^{-1}I_{\mathbb{Z}_p}\mathbb{S}$ in Section 1.

Although phrased in terms of the cyclotomic trace, the Main Theorem has a purely K -theoretic interpretation. Hesselholt-Madsen [26] shows that the completion map $TC(\mathbb{Z})_p^\wedge \rightarrow TC(\mathbb{Z}_p)_p^\wedge$ is a weak equivalence and the cyclotomic trace $K(\mathbb{Z}_p)_p^\wedge \rightarrow TC(\mathbb{Z}_p)_p^\wedge$ is a connective cover. It follows that the connective cover of the homotopy fiber of the cyclotomic trace is weakly equivalent to the homotopy fiber of the completion map $K(\mathbb{Z})_p^\wedge \rightarrow K(\mathbb{Z}_p)_p^\wedge$. Quillen's localization sequence for $\mathbb{Z} \rightarrow \mathbb{Z}[\frac{1}{p}]$ and $\mathbb{Z}_p \rightarrow \mathbb{Q}_p$,

$$\begin{array}{ccccccc} K(\mathbb{Z}/p) & \longrightarrow & K(\mathbb{Z}) & \longrightarrow & K(\mathbb{Z}[1/p]) & \longrightarrow & \Sigma \cdots \\ \text{id} \downarrow & & \downarrow & & \downarrow & & \\ K(\mathbb{Z}/p) & \longrightarrow & K(\mathbb{Z}_p) & \longrightarrow & K(\mathbb{Q}_p) & \longrightarrow & \Sigma \cdots \end{array}$$

displays the inner square as a homotopy pullback square, which implies that the homotopy fiber of the completion map $K(\mathbb{Z}) \rightarrow K(\mathbb{Z}_p)$ is weakly equivalent to the homotopy fiber of the completion map $K(\mathbb{Z}[\frac{1}{p}]) \rightarrow K(\mathbb{Q}_p)$. The (affirmed) Quillen-Lichtenbaum conjecture [45, VI.8.2] then implies that the maps

$$K(\mathbb{Z}[\frac{1}{p}])_p^\wedge \longrightarrow L_{K(1)}K(\mathbb{Z}[\frac{1}{p}]), \quad K(\mathbb{Q}_p)_p^\wedge \longrightarrow L_{K(1)}K(\mathbb{Q}_p),$$

are weak equivalences after taking 1-connected covers. Looking in low dimensions, $K(\mathbb{Z}[\frac{1}{p}])_p^\wedge \rightarrow L_{K(1)}K(\mathbb{Z}[\frac{1}{p}])$ is actually a connective cover, while $K(\mathbb{Q}_p)_p^\wedge \rightarrow L_{K(1)}K(\mathbb{Q}_p)$ induces an isomorphism on homotopy groups in degrees > 0 and an injection in degree 0. It then follows that the (connective) homotopy fiber of $K(\mathbb{Z}[\frac{1}{p}])_p^\wedge \rightarrow K(\mathbb{Q}_p)_p^\wedge$ is weakly equivalent to the connective cover of the homotopy fiber of $L_{K(1)}K(\mathbb{Z}[\frac{1}{p}]) \rightarrow L_{K(1)}K(\mathbb{Q}_p)$.

The identification of this homotopy fiber is a consequence of a general K -theoretic duality theorem for number fields related to Tate-Potou duality. Tate-Poitou duality describes the relationship between the étale cohomology of S -integers in number fields and their completions in terms of a long exact sequence where the third term is a Pontryagin dual related to the first term. In the most basic case,

for $p > 2$ a prime in \mathbb{Z} and a number field F , we get a long exact sequence

$$\begin{aligned}
0 \rightarrow H_{\text{ét}}^0(\mathcal{O}_F[\frac{1}{p}]; \mathbb{Z}_p(k)) &\rightarrow \prod_{\nu|p} H_{\text{ét}}^0(F_\nu; \mathbb{Z}_p(k)) \rightarrow (H_{\text{ét}}^2(\mathcal{O}_F[\frac{1}{p}], \mathbb{Q}_p/\mathbb{Z}_p(1-k)))^* \\
&\searrow \\
&\hookrightarrow H_{\text{ét}}^1(\mathcal{O}_F[\frac{1}{p}]; \mathbb{Z}_p(k)) \rightarrow \prod_{\nu|p} H_{\text{ét}}^1(F_\nu; \mathbb{Z}_p(k)) \rightarrow (H_{\text{ét}}^1(\mathcal{O}_F[\frac{1}{p}], \mathbb{Q}_p/\mathbb{Z}_p(1-k)))^* \\
&\searrow \\
&\hookrightarrow H_{\text{ét}}^2(\mathcal{O}_F[\frac{1}{p}]; \mathbb{Z}_p(k)) \rightarrow \prod_{\nu|p} H_{\text{ét}}^2(F_\nu; \mathbb{Z}_p(k)) \rightarrow (H_{\text{ét}}^0(\mathcal{O}_F[\frac{1}{p}], \mathbb{Q}_p/\mathbb{Z}_p(1-k)))^* \rightarrow 0
\end{aligned}$$

where $H_{\text{ét}}^*(-; \mathbb{Z}_p(k))$ denotes Jannsen's continuous étale cohomology, \mathcal{O}_F denotes the ring of integers, F_ν denotes completion at the prime ν , and $(-)^*$ denotes Pontryagin dual. The Main Theorem is a special case of a spectrum-level K -theoretic version of Tate-Poitou duality encoding the behavior of the completion map in the algebraic K -theory of rings of integers in number fields.

Thomason's work [43] on the Quillen-Lichtenbaum conjecture relates the étale cohomology groups in the above sequence to the homotopy groups of the $K(1)$ -localization of algebraic K -theory spectra. Specifically, Thomason's descent spectral sequence [43, Theorem 4.1] (see also [*ibid.*, App. A]) gives canonical isomorphisms

$$\pi_n(L_{K(1)}K(R)) \cong \begin{cases} H_{\text{ét}}^0(R; \mathbb{Z}_p(\frac{n}{2})) \oplus H_{\text{ét}}^2(R; \mathbb{Z}_p(\frac{n}{2} + 1)) & n \text{ even} \\ H_{\text{ét}}^1(R; \mathbb{Z}_p(\frac{n+1}{2})) & n \text{ odd} \end{cases}$$

for $R = \mathcal{O}_F[1/p]$ or F_ν . Letting $M_{\mathbb{Q}_p/\mathbb{Z}_p}$ denote the Moore spectrum for $\mathbb{Q}_p/\mathbb{Z}_p$, Thomason's descent spectral sequence also gives non-canonical isomorphisms

$$\pi_n(L_{K(1)}K(R) \wedge M_{\mathbb{Q}_p/\mathbb{Z}_p}) \cong \begin{cases} H_{\text{ét}}^0(R; \mathbb{Q}_p/\mathbb{Z}_p(\frac{n}{2})) \oplus H_{\text{ét}}^2(R; \mathbb{Q}_p/\mathbb{Z}_p(\frac{n}{2} + 1)) & n \text{ even} \\ H_{\text{ét}}^1(R; \mathbb{Q}_p/\mathbb{Z}_p(\frac{n+1}{2})) & n \text{ odd} \end{cases}$$

for $R = \mathcal{O}_F[1/p]$. Algebraically, we can then use these isomorphisms to rewrite the Tate-Poitou sequence (non-canonically) as the long exact sequence

$$\begin{aligned}
&\cdots \rightarrow (\pi_{-1-n}(L_{K(1)}K(\mathcal{O}_F[1/p]) \wedge M_{\mathbb{Q}_p/\mathbb{Z}_p}))^* \\
&\hookrightarrow \pi_n L_{K(1)}K(\mathcal{O}_F[1/p]) \rightarrow \prod \pi_n L_{K(1)}K(F_\nu) \rightarrow \cdots
\end{aligned}$$

with the first term (on the left) the homotopy groups of the $K(1)$ -localization of $K(\mathcal{O}_F[1/p])$ and the second term the homotopy groups of the product of the $K(1)$ -localizations of the K -theory of the completed fields. We can interpret the third term as the homotopy groups of a spectrum as well, using Brown-Comenetz or Anderson duality:

$$\begin{aligned}
\pi_{-1-n}(L_{K(1)}K(\mathcal{O}_F[1/p]) \wedge M_{\mathbb{Q}_p/\mathbb{Z}_p}))^* &\cong \pi_n(\Sigma^{-1}I_{\mathbb{Q}/\mathbb{Z}}(L_{K(1)}K(\mathbb{Z}) \wedge M_{\mathbb{Q}_p/\mathbb{Z}_p})) \\
&\cong \pi_n(\Sigma^{-1}I_{\mathbb{Z}_p}L_{K(1)}K(\mathbb{Z})),
\end{aligned}$$

where as above $I_{\mathbb{Q}/\mathbb{Z}}$ denotes the Brown-Comenetz dual and $I_{\mathbb{Z}_p}$ denotes the \mathbb{Z}_p -Anderson dual. The following theorem lifts this exact sequence to a cofiber sequence on the spectrum level and makes the maps canonical.

Theorem (*K-Theoretic Tate-Poitou Duality*). *Let $p > 2$ be a prime number. Let F be a number field, \mathcal{O}_F its ring of integers, and S the set of primes of \mathcal{O}_F above p . For $\nu \in S$, write F_ν for the ν -completion of F . The homotopy fiber $\text{Fib}(\kappa)$ of the completion map in $K(1)$ -local algebraic K -theory*

$$\kappa: L_{K(1)}K(\mathcal{O}_F[\frac{1}{p}]) \longrightarrow \prod_{\nu \in S} L_{K(1)}K(F_\nu)$$

is weakly equivalent to

$$\Sigma^{-1}I_{\mathbb{Q}/\mathbb{Z}}(L_{K(1)}K(\mathcal{O}_F[\frac{1}{p}]) \wedge M_{\mathbb{Q}_p/\mathbb{Z}_p}) \simeq \Sigma^{-1}I_{\mathbb{Z}_p}L_{K(1)}K(\mathcal{O}_F[\frac{1}{p}]).$$

The weak equivalence $\text{Fib}(\kappa) \rightarrow \Sigma^{-1}I_{\mathbb{Z}_p}L_{K(1)}K(\mathcal{O}_F[\frac{1}{p}])$ is adjoint to the map

$$L_{K(1)}K(\mathcal{O}_F[\frac{1}{p}]) \wedge \text{Fib}(\kappa) \longrightarrow \Sigma^{-1}I_{\mathbb{Z}_p}\mathbb{S}$$

induced by the $L_{K(1)}K(\mathcal{O}_F[\frac{1}{p}])$ -module structure map $L_{K(1)}K(\mathcal{O}_F[\frac{1}{p}]) \wedge \text{Fib}(\kappa) \rightarrow \text{Fib}(\kappa)$ and a canonical map

$$\text{Fib}(\kappa) \longrightarrow \Sigma^{-1}I_{\mathbb{Z}_p}\mathbb{S}.$$

The previous theorem establishes a global arithmetic duality for algebraic K -theory. Clausen's MIT thesis [14] contains work in this direction; in particular, Clausen produces a duality map of a similar type to the one in the previous theorem (with different details), presumably related through Gross-Hopkins duality. Work in progress of Schlank and Stojanoska [40] establishes arithmetic duality results for a much wider range of theories.

The previous theorem has interesting consequences for work of Calegari and Emerton [11, 13, 12, 21] on the completed (co)homology of congruence subgroups of a number field (as part of the study of the p -adic Langlands program). The paper [11] specifically relates the completed cohomology of congruence subgroups to the homotopy fiber of the completion map on algebraic K -theory. The K -theoretic Tate-Poitou duality theorem above implies in the affirmative a conjecture of Calegari [11, 1.5] regarding the homotopy groups of this fiber; the affirmed conjecture sharpens some of his explicit calculations and also shows that the fiber of the cyclotomic trace can be identified as a spectrum-level lift of the completed cohomology. We discuss this further in Section 4.

We stated the consequences of the K -theoretic Tate-Poitou duality theorem for the homotopy fiber of the cyclotomic trace for \mathbb{S} and therefore for \mathbb{Z} above. In fact, most of the discussion obtains more generally to identify the homotopy fiber of the cyclotomic trace for a number ring \mathcal{O}_F . As above, let S denote the set of primes of \mathcal{O}_F lying above p and for each $\nu \in S$, write R_ν for the completion of \mathcal{O}_F . We have Quillen's localization sequence for $\mathcal{O}_F \rightarrow \mathcal{O}_F[1/p]$ and $\prod R_\nu \rightarrow \prod F_\nu$,

$$\begin{array}{ccccccc} \bigvee_{\nu \in S} K(\mathcal{O}_F/\nu) & \longrightarrow & K(\mathcal{O}_F) & \longrightarrow & K(\mathcal{O}_F[\frac{1}{p}]) & \longrightarrow & \Sigma \cdots \\ \downarrow \cong & & \downarrow & & \downarrow & & \\ \bigvee_{\nu \in S} K(R_\nu/\nu) & \longrightarrow & K(\prod_{\nu \in S} R_\nu) & \longrightarrow & K(\prod_{\nu \in S} F_\nu) & \longrightarrow & \Sigma \cdots, \end{array}$$

which identifies the homotopy fiber of the completion map $K(\mathcal{O}_F) \rightarrow K(\prod R_\nu)$ as weakly equivalent to the homotopy fiber of the completion map $K(\mathcal{O}_F[1/p]) \rightarrow K(\prod F_\nu)$. Each \mathcal{O}_F/ν is a finite field of characteristic p , and so $K(\mathcal{O}_F/\nu)_p^\wedge$ is an

Eilenberg-Mac Lane spectrum. The Quillen localization sequence then implies that the map $L_{K(1)}K(\mathcal{O}_F) \rightarrow L_{K(1)}K(\mathcal{O}_F[1/p])$ is a weak equivalence. Since

$$\prod R_\nu \cong (\mathcal{O}_F)_p^\wedge \cong \mathcal{O}_F \otimes \mathbb{Z}_p,$$

Hesselholt-Madsen [26, Add. 6.2] shows that the map $TC(\mathcal{O}_F)_p^\wedge \rightarrow TC(\prod R_\nu)_p^\wedge$ is a weak equivalence. Hesselholt-Madsen [26, Theorem D] shows that the map $K(\prod R_\nu)_p^\wedge \rightarrow TC(\prod R_\nu)_p^\wedge$ is a connective cover. Applying the K -theoretic Tate-Poitou theorem above and the (affirmed) Quillen-Lichtenbaum conjecture, we obtain the following corollary.

Corollary. *Let F be a number field and let $p \in \mathbb{Z}$ be an odd prime. Then there is a canonical map in the stable category from the homotopy fiber of the cyclotomic trace $K(\mathcal{O}_F)_p^\wedge \rightarrow TC(\mathcal{O}_F)_p^\wedge$ to $\Sigma^{-1}I_{\mathbb{Z}_p}(L_{K(1)}K(\mathcal{O}_F))$ that induces an isomorphism on homotopy groups in dimensions ≥ 2 .*

We have more to say on the fiber of the cyclotomic trace $K(\mathbb{S})_p^\wedge \rightarrow TC(\mathbb{S})_p^\wedge$, which we have put in Sections 2–3, after we discuss the construction of the canonical map $\text{Fib}(\kappa) \rightarrow \Sigma^{-1}I_{\mathbb{Z}_p}\mathbb{S}$ in Section 1. We also include more details on the connection to the completed homology of congruence subgroups in Section 4. The proof of the K -theoretic Tate-Poitou duality theorem occupies Sections 5–8. The argument recasts $\text{Fib}(\kappa)$, the homotopy fiber of the localized completion map, as étale hypercohomology of an appropriate presheaf of symmetric spectra and compares (via a version of Thomason’s descent spectral sequence [43, 4.1]) the $K(1)$ -localized K -theory pairing with the cup product pairing in étale hypercohomology that induces classical Tate-Poitou duality. Section 5 provides the preliminaries needed on the K -theory presheaf and reviews some aspects of the hypercohomology of presheaves of symmetric spectra. Section 6 does the actual comparison of $\text{Fib}(\kappa)$ with the hypercohomology spectrum and Section 7 then proves the theorem. Section 8 is a technical section that constructs a functorial monoidal version of the Whitehead tower used in Section 7 to compare the pairings. The final section proves a folk theorem used in Section 1.

Conventions. We describe here our general conventions for working in the stable category throughout the paper. Later sections involve presheaves of symmetric spectra, and we set out more conventions for working with presheaves and the K -theory functor in Section 5.

Except in the introduction (above), we write “ \wedge ” for the point-set smash product of symmetric spectra and “ \wedge^L ” for the derived smash product in the stable category, partly as a reminder that we are working in the stable category. In the introduction, all occurrences of “ \wedge ” denote the derived smash product. Throughout the paper, we write $F(X, -)$ for the right adjoint of the derived smash product $(-) \wedge X$; we have no call to use the corresponding point-set functor in this paper.

Some of the calculations in Sections 1–3 involves precise accounting for some difficult signs. For this we use the following conventions: Suspension is $(-) \wedge S^1$ and cone is $(-) \wedge I$, where in the latter case we use 1 as the basepoint. Cofiber sequences are sequences isomorphic (in the stable category) to Puppe sequences formed in the usual way using this suspension and cone. A cofiber sequence leads to a long exact sequence of homotopy groups; we use the sign convention that for a map $f: A \rightarrow B$, the connecting map $\pi_n C f \rightarrow \pi_{n-1} A$ in the long exact sequence of homotopy groups is $(-1)^n \sigma_n^{-1}$ composed with the Puppe sequence map

$\pi_n C f \rightarrow \pi_n \Sigma A$, where σ_n denotes the suspension isomorphism $\pi_{n-1} A \rightarrow \pi_n \Sigma A$. (Consideration of the example of standard cells explains the desirability of the sign.)

We form the homotopy fiber $\text{Fib}(f)$ of a map f using the space of paths starting from the basepoint; we then have a canonical map $\Sigma \text{Fib}(f) \rightarrow C f$ in the usual way (using the suspension coordinate to follow the path and then follow the cone). We switch between fiber sequences and cofiber sequences at will, using the convention that for the fiber sequence

$$\Omega B \xrightarrow{\delta} \text{Fib}(f) \xrightarrow{\phi} A \xrightarrow{f} B,$$

the sequence

$$\text{Fib}(f) \xrightarrow{\phi} A \xrightarrow{f} B \xrightarrow{-\Sigma \delta \circ \epsilon^{-1}} \Sigma \text{Fib}(f)$$

is a cofiber sequence where $\epsilon: \Sigma \Omega B \rightarrow B$ is the counit of the Σ/Ω adjunction. For the long exact sequence of homotopy groups associated to a fiber sequence, we use the long exact sequence of homotopy groups of the associated cofiber sequence. In terms of the fiber sequence displayed above, the connecting map $\pi_{n+1} B \rightarrow \pi_n \text{Fib}(f)$ is the composite of the canonical isomorphism $\pi_{n+1} B \cong \pi_n \Omega B$ and the map $(-1)^n \pi_n \delta$.

For a cofiber sequence

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$$

and a fixed spectrum X , the sequence

$$\Omega F(A, X) \xrightarrow{-h^*} F(C, X) \xrightarrow{g^*} F(B, X) \xrightarrow{f^*} F(A, X)$$

is a fiber sequence and

$$F(C, X) \xrightarrow{g^*} F(B, X) \xrightarrow{f^*} F(A, X) \xrightarrow{h^*} \Sigma F(C, X)$$

is a cofiber sequence.

Acknowledgments. The authors thank Bill Dwyer, Mike Hopkins, Lars Hesselholt, John Rognes, and Matthias Strauch for helpful conversations, and the Hausdorff Research Institute for Mathematics for its hospitality while some of this work was done.

1. K -THEORETIC LOCAL DUALITY AND THE CONSTRUCTION OF THE CANONICAL MAP $\text{Fib}(\kappa) \rightarrow \Sigma^{-1} I_{\mathbb{Z}_p} \mathbb{S}$

The Tate-Poitou duality theorem in global arithmetic derives from a much easier local duality theorem. We have a corresponding K -theoretic local duality theorem, which is also much easier at least in the sense that it requires no new tools to prove (and is well-known to experts). We state and prove the K -theoretic local duality theorem in this section, deducing it from the local duality theorem in arithmetic. The argument is parallel to the argument used in Sections 7 to prove the K -theoretic Tate-Poitou duality theorem and explains the construction of the canonical map $\text{Fib}(\kappa) \rightarrow \Sigma^{-1} I_{\mathbb{Z}_p} \mathbb{S}$, which is characterized by its relationship to a corresponding map $L_{K(1)} K(\mathbb{Q}_p) \rightarrow I_{\mathbb{Z}_p} \mathbb{S}$ in the K -theoretic local duality of \mathbb{Q}_p .

In arithmetic, local duality is an isomorphism

$$H_{\text{ét}}^i(k; M) \xrightarrow{\cong} (H_{\text{ét}}^{2-i}(k; M^*(1)))^*$$

where k is the field of fractions of a complete discrete valuation ring whose residue field is finite (e.g., a finite extension of \mathbb{Q}_p), M is a finite Galois module, and $(-)^*$ denotes the Pontryagin dual. The map is induced by the cup product pairing

$$H_{\text{ét}}^i(k; M) \otimes H_{\text{ét}}^{2-i}(k; M^*(1)) \longrightarrow H_{\text{ét}}^2(k; M \otimes M^*(1)) \longrightarrow H_{\text{ét}}^2(k; \mathbb{Q}/\mathbb{Z}(1))$$

and a canonical map

$$H_{\text{ét}}^2(k; \mathbb{Q}/\mathbb{Z}(1)) \xrightarrow{\cong} H_{\text{ét}}^2(k; \mathbb{G}_m) \xrightarrow{\cong} \mathbb{Q}/\mathbb{Z}$$

given by the Hasse invariant. Letting $M = \mathbb{Z}/p^n(j)$ and taking the limit $n \rightarrow \infty$, we get an isomorphism

$$H_{\text{ét}}^i(k; \mathbb{Z}_p(j)) \cong (H_{\text{ét}}^{2-i}(k; \mathbb{Q}_p/\mathbb{Z}_p(1-j)))^*$$

(where the group on the left is Jannsen's continuous étale cohomology).

For K -theory, the E_∞ multiplication induces a map

$$L_{K(1)}K(k) \wedge^L L_{K(1)}K(k) \wedge^L M_{\mathbb{Q}_p/\mathbb{Z}_p} \longrightarrow L_{K(1)}K(k) \wedge^L M_{\mathbb{Q}_p/\mathbb{Z}_p}$$

and we have a canonical map

$$(1.1) \quad L_{K(1)}K(k) \wedge^L M_{\mathbb{Q}_p/\mathbb{Z}_p} \longrightarrow I_{\mathbb{Q}/\mathbb{Z}}\mathbb{S}$$

essentially induced by the Hasse invariant as follows. Such a map is uniquely determined by specifying a homomorphism

$$\pi_0(L_{K(1)}K(k) \wedge^L M_{\mathbb{Q}_p/\mathbb{Z}_p}) \longrightarrow \mathbb{Q}/\mathbb{Z}.$$

Thomason's descent spectral sequence puts $\pi_0(L_{K(1)}K(k) \wedge^L M_{\mathbb{Q}_p/\mathbb{Z}_p})$ into a canonical short exact sequence

$$0 \longrightarrow H_{\text{ét}}^2(k; \mathbb{Q}_p/\mathbb{Z}_p(1)) \longrightarrow \pi_0(L_{K(1)}K(k) \wedge^L M_{\mathbb{Q}_p/\mathbb{Z}_p}) \longrightarrow H_{\text{ét}}^0(k; \mathbb{Q}_p/\mathbb{Z}_p) \longrightarrow 0$$

which is canonically split by the map $\mathbb{Q}_p/\mathbb{Z}_p = H_{\text{ét}}^0(k; \mathbb{Q}_p/\mathbb{Z}_p) \rightarrow \pi_0(L_{K(1)}K(k) \wedge^L M_{\mathbb{Q}_p/\mathbb{Z}_p})$ induced by the unit $\mathbb{S} \rightarrow K(k)$. Combining this with the Hasse invariant gives a canonical map

$$\pi_0(L_{K(1)}K(k) \wedge^L M_{\mathbb{Q}_p/\mathbb{Z}_p}) \longrightarrow H_{\text{ét}}^2(k; \mathbb{Q}_p/\mathbb{Z}_p(1)) \longrightarrow H_{\text{ét}}^2(k; \mathbb{G}_m) \longrightarrow \mathbb{Q}/\mathbb{Z}.$$

Theorem 1.2 (*K-Theoretic Local Duality*). *Let k be the field of fractions of a complete discrete valuation ring whose residue field is finite. The map*

$$L_{K(1)}K(k) \longrightarrow I_{\mathbb{Q}/\mathbb{Z}}(L_{K(1)}K(k) \wedge^L M_{\mathbb{Q}_p/\mathbb{Z}_p}) \simeq I_{\mathbb{Z}_p}(L_{K(1)}K(k))$$

adjoint to the composite map

$$L_{K(1)}K(k) \wedge^L L_{K(1)}K(k) \wedge^L M_{\mathbb{Q}_p/\mathbb{Z}_p} \longrightarrow I_{\mathbb{Q}/\mathbb{Z}}\mathbb{S}$$

described above is a weak equivalence.

Proof. Because $L_{K(1)}K(k)$ and $I_{\mathbb{Z}_p}(L_{K(1)}K(k))$ are both p -complete, it suffices to check that the map is a weak equivalence after taking the derived smash product with the mod p Moore spectrum M_p , or equivalently taking the homotopy cofiber of multiplication by p . Using the canonical weak equivalence

$$I_{\mathbb{Q}/\mathbb{Z}}(L_{K(1)}K(k) \wedge^L M_{\mathbb{Q}_p/\mathbb{Z}_p})/p \simeq I_{\mathbb{Q}/\mathbb{Z}}(L_{K(1)}K(k)/p),$$

the induced map

$$L_{K(1)}K(k)/p \longrightarrow I_{\mathbb{Q}/\mathbb{Z}}(L_{K(1)}K(k)/p)$$

is adjoint to the composite map

$$L_{K(1)}K(k)/p \wedge^L L_{K(1)}K(k)/p \longrightarrow L_{K(1)}K(k)/p \longrightarrow L_{K(1)}K(k) \wedge^L M_{\mathbb{Q}_p/\mathbb{Z}_p} \longrightarrow I_{\mathbb{Q}/\mathbb{Z}}\mathbb{S}$$

induced by the E_∞ pairing on $L_{K(1)}K(k)$ and the usual pairing $M_p \wedge^L M_p \rightarrow M_p$ for the first map and by the usual inclusion of M_p in $M_{\mathbb{Q}_p/\mathbb{Z}_p}$ for the second map. Thus, it suffices to check that the pairing above induces a perfect pairing

$$\pi_q(L_{K(1)}K(k)/p) \otimes \pi_{-q}(L_{K(1)}K(k)/p) \longrightarrow \mathbb{Q}/\mathbb{Z}.$$

It is folklore that Thomason's descent spectral sequence [43, 4.1] is multiplicative with the multiplication on the E_2 -term induced by the cup product in étale cohomology; we give a proof of this in Theorem 7.2 below (see also Remark 7.8). We therefore have a perfect pairing on the $E_2 = E_\infty$ -term by local duality in arithmetic, and it follows that we have a perfect pairing on homotopy groups (cf. Proposition 7.7). \square

The proof of the global K -theoretic Tate-Poitou duality theorem in Section 7 follows the same general outline as the proof of the local duality theorem above, except instead of using the multiplication on a single $K(1)$ -localized K -theory spectrum, we study the pairing of $L_{K(1)}K(\mathcal{O}_F[1/p])$ with $\text{Fib}(\kappa)$. The proof above studies the pairing in terms of étale cohomology and the first issue in the global case is finding an (up to filtration) étale cohomological interpretation of $\pi_* \text{Fib}(\kappa)$; we do this in Section 6. While local duality is usually stated in terms of the cup product (as reviewed above), the global case is most conveniently written in terms of the Yoneda product on Ext . The remainder of the proof of the global case in Section 7 fills in the technical details of the pairing on Thomason's descent spectral sequence and identifies the pairing on the E_2 -term both in terms of the cup product (in general) and in terms of the Yoneda product for the specific pairing in the global case.

In local duality, the map $\pi_0(L_{K(1)}K(k) \wedge^L M_{\mathbb{Q}_p/\mathbb{Z}_p}) \rightarrow \mathbb{Q}/\mathbb{Z}$ comes from the Hasse invariant isomorphism $H_{\text{ét}}^2(k; \mathbb{Q}/\mathbb{Z}(1)) \cong \mathbb{Q}/\mathbb{Z}$, or in terms of p -torsion, the isomorphism $H_{\text{ét}}^2(k; \mathbb{Q}_p/\mathbb{Z}_p(1)) \cong \mathbb{Q}_p/\mathbb{Z}_p$. For global duality, the map is related to the Albert-Brauer-Hasse-Noether sequence for $\mathcal{O}_F[1/p]$: the p -torsion version of this sequence takes the form of an exact sequence

$$0 \longrightarrow H_{\text{ét}}^2(\mathcal{O}_F[1/p]; \mathbb{Q}_p/\mathbb{Z}_p(1)) \longrightarrow \prod_{\nu \in S} H_{\text{ét}}^2(F_\nu; \mathbb{Q}_p/\mathbb{Z}_p(1)) \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow 0$$

where S is the set of primes lying above p and $p > 2$. Looking at the map of short exact sequences from Thomason's descent spectral sequence

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ H_{\text{ét}}^2(\mathcal{O}_F[1/p]; \mathbb{Q}_p/\mathbb{Z}_p(1)) & \longrightarrow & \prod H_{\text{ét}}^2(F_\nu; \mathbb{Q}_p/\mathbb{Z}_p(1)) \\ \downarrow & & \downarrow \\ \pi_0(L_{K(1)}K(\mathcal{O}_F[1/p]) \wedge^L M_{\mathbb{Q}_p/\mathbb{Z}_p}) & \longrightarrow & \prod \pi_0(L_{K(1)}K(F_\nu) \wedge^L M_{\mathbb{Q}_p/\mathbb{Z}_p}) \\ \downarrow & & \downarrow \\ H_{\text{ét}}^0(\mathcal{O}_F[1/p]; \mathbb{Q}_p/\mathbb{Z}_p) & \longrightarrow & \prod H_{\text{ét}}^0(F_\nu; \mathbb{Q}_p/\mathbb{Z}_p) \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

we get a canonical map from $\mathbb{Q}_p/\mathbb{Z}_p$ to the cokernel

$$C = \text{coker}(\pi_0(L_{K(1)}K(\mathcal{O}_F[1/p]) \wedge^L M_{\mathbb{Q}_p/\mathbb{Z}_p}) \longrightarrow \pi_0(\prod L_{K(1)}K(F_\nu) \wedge^L M_{\mathbb{Q}_p/\mathbb{Z}_p})).$$

This and the long sequence on homotopy groups gives us a canonical map from $\mathbb{Q}_p/\mathbb{Z}_p$ to $\pi_{-1}(\text{Fib}(\kappa) \wedge^L M_{\mathbb{Q}_p/\mathbb{Z}_p})$.

Theorem 1.3. *The canonical map $\mathbb{Q}_p/\mathbb{Z}_p \rightarrow \pi_{-1}(\mathrm{Fib}(\kappa) \wedge^L M_{\mathbb{Q}_p/\mathbb{Z}_p})$ is a split injection and has a unique retraction that commutes with the K -theory transfer associated to inclusions of number fields.*

Proof. First we note that the K -theory transfer associated to the inclusion of number fields extends to a well-defined map in the stable category on $\mathrm{Fib}(\kappa)$: For $F \subset E$ an inclusion of number fields, $\mathcal{O}_E[1/p]$ is a finitely generated free $\mathcal{O}_F[1/p]$ -module and we have an associated K -theory transfer map $K(\mathcal{O}_E[1/p]) \rightarrow K(\mathcal{O}_F[1/p])$ induced by regarding a finitely generated projective $\mathcal{O}_E[1/p]$ -module as a finitely generated projective $\mathcal{O}_F[1/p]$ -module. For the p -completions

$$\mathcal{O}_E \otimes \mathbb{Q}_p \cong \prod_{\nu \in S_E} E_\nu, \quad \mathcal{O}_F \otimes \mathbb{Q}_p \cong \prod_{\nu \in S_F} F_\nu,$$

we have an associated K -theory transfer map and in the standard models for K -theory (see Section 5), the diagram

$$\begin{array}{ccc} K(\mathcal{O}_E[1/p]) & \longrightarrow & K(\mathcal{O}_E \otimes \mathbb{Q}_p) \\ \downarrow & & \downarrow \\ K(\mathcal{O}_F[1/p]) & \longrightarrow & K(\mathcal{O}_F \otimes \mathbb{Q}_p) \end{array}$$

commutes up to canonical homotopy since for a finitely generated projective $\mathcal{O}_E[1/p]$ -module P , the underlying $(\mathcal{O}_F \otimes \mathbb{Q}_p)$ -module of $P \otimes \mathbb{Q}_p$ is canonically isomorphic to the underlying $\mathcal{O}_F[1/p]$ -module of P tensored with \mathbb{Q}_p . This is enough structure to specify a canonical map in the stable category on the homotopy fibers.

Uniqueness is clear because in the case $F = \mathbb{Q}$, the inclusion of $\mathbb{Q}_p/\mathbb{Z}_p$ in $\pi_{-1}(\mathrm{Fib}(\kappa) \wedge^L M_{\mathbb{Q}_p/\mathbb{Z}_p})$ is an isomorphism. To see this note that the inclusion of $\mathbb{Q}_p/\mathbb{Z}_p$ in the cokernel C is surjective (because the map $H^0(\mathbb{Z}[1/p]; \mathbb{Q}_p/\mathbb{Z}_p) \rightarrow H^0(\mathbb{Q}_p; \mathbb{Q}_p/\mathbb{Z}_p)$ is an isomorphism) and the map from the cokernel C to $\pi_{-1}(\mathrm{Fib}(\kappa) \wedge^L M_{\mathbb{Q}_p/\mathbb{Z}_p})$ is surjective (because the map $H^1(\mathbb{Z}[1/p]; \mathbb{Q}_p/\mathbb{Z}_p) \rightarrow H^1(\mathbb{Q}_p; \mathbb{Q}_p/\mathbb{Z}_p)$ is surjective due to abelianized Galois group considerations, or see Corollary 2.6 below).

For existence of the splitting, since \mathbb{Q} is initial among number fields, we just need to know that for an inclusion of number fields $F \subset E$, the diagram

$$\begin{array}{ccc} \mathbb{Q}_p/\mathbb{Z}_p & \longrightarrow & C_E \\ \mathrm{id} \downarrow & & \downarrow \\ \mathbb{Q}_p/\mathbb{Z}_p & \longrightarrow & C_F \end{array}$$

commutes where C_E and C_F are the cokernels C associated to E and F above and the map is induced by the K -theory transfer. Because $H_{\mathrm{\acute{e}t}}^2(F_\nu; \mathbb{Q}_p/\mathbb{Z}_p(1))$ is the p -torsion in $H_{\mathrm{\acute{e}t}}^2(F_\nu; \mathbb{G}_m)$, the basic properties of a class formation (q.v. Proposition 1(ii) in [42, XI§2]) imply that it is enough to see that the diagram

$$\begin{array}{ccc} H_{\mathrm{\acute{e}t}}^2(\mathcal{O}_E \otimes \mathbb{Q}_p; \mathbb{Q}_p/\mathbb{Z}_p(1)) & \longrightarrow & \pi_0(L_{K(1)}(\mathcal{O}_E \otimes \mathbb{Q}_p) \wedge^L M_{\mathbb{Q}_p/\mathbb{Z}_p}) \\ \downarrow & & \downarrow \\ H_{\mathrm{\acute{e}t}}^2(\mathcal{O}_F \otimes \mathbb{Q}_p; \mathbb{Q}_p/\mathbb{Z}_p(1)) & \longrightarrow & \pi_0(L_{K(1)}(\mathcal{O}_F \otimes \mathbb{Q}_p) \wedge^L M_{\mathbb{Q}_p/\mathbb{Z}_p}) \end{array}$$

commutes where the left vertical map is the transfer in étale cohomology. This follows from the folklore theorem that the K -theory transfer for Galois extensions

induces the étale cohomology transfer on the E_2 -page of Thomason's descent spectral sequence; see Theorem 9.1. \square

Applying the previous theorem, the composite of the canonical map $\pi_{-1}(\text{Fib}(\kappa) \wedge^L M_{\mathbb{Q}_p/\mathbb{Z}_p}) \rightarrow \mathbb{Q}_p/\mathbb{Z}_p$ with the inclusion of p -torsion $\mathbb{Q}_p/\mathbb{Z}_p \rightarrow \mathbb{Q}/\mathbb{Z}$ specifies a canonical map

$$(1.4) \quad \text{Fib}(\kappa) \wedge^L M_{\mathbb{Q}_p/\mathbb{Z}_p} \longrightarrow I_{\mathbb{Q}/\mathbb{Z}}\mathbb{S}.$$

Adjoint to this map is the canonical map

$$(1.5) \quad u_{\mathcal{O}_F} : \text{Fib}(\kappa) \longrightarrow F(M_{\mathbb{Z}_p}, I_{\mathbb{Q}/\mathbb{Z}}\mathbb{S}) \simeq F(\Sigma^{-1}M_{\mathbb{Z}_p}, \Sigma^{-1}I_{\mathbb{Q}/\mathbb{Z}}\mathbb{S}) = \Sigma^{-1}I_{\mathbb{Z}_p}\mathbb{S}$$

in the statement of the Main Theorem and the K -theoretic Tate-Poitou duality theorem. In particular, we have constructed u to be compatible with the corresponding map

$$(1.6) \quad v_{F_\nu} : L_{K(1)}K(F_\nu) \longrightarrow I_{\mathbb{Z}_p}\mathbb{S}$$

for local duality adjoint to the map (1.1) (for $k = F_\nu$). They are compatible in the sense that v_{F_ν} is the composite

$$L_{K(1)}K(F_\nu) \longrightarrow \Sigma \text{Fib}(\kappa) \xrightarrow{\Sigma u_{\mathcal{O}_F}} \Sigma \Sigma^{-1}I_{\mathbb{Z}_p}\mathbb{S} \cong I_{\mathbb{Z}_p}\mathbb{S},$$

where the first map is a component of the map

$$\prod_{\nu \in S} L_{K(1)}K(F_\nu) \longrightarrow \Sigma \text{Fib}(\kappa)$$

in the cofiber sequence (associated to the fiber sequence) defining $\text{Fib}(\kappa)$.

2. THE HOMOTOPY FIBER OF THE CYCLOTOMIC TRACE

In principle, the Main Theorem identifies the homotopy type of the homotopy fiber of the cyclotomic trace, but in general, the homotopy type of $L_{K(1)}K(\mathbb{Z})$ is not well-understood. Nevertheless, combining results of Dwyer-Mitchell [19], [7] with the work above (and the work in [6, §5]), we get a pretty good picture of the cofiber sequence

$$(2.1) \quad \text{Fib}(\tau) \longrightarrow K(\mathbb{Z})_p^\wedge \longrightarrow TC(\mathbb{Z})_p^\wedge \longrightarrow \Sigma \text{Fib}(\tau)$$

for all odd primes and a very good picture of it for odd primes that satisfy the Kummer-Vandiver condition (see below). By the affirmed Quillen-Lichtenbaum conjecture, little is lost by working with the $K(1)$ -local sequence

$$(2.2) \quad \text{Fib}(\kappa) \longrightarrow L_{K(1)}K(\mathbb{Z}) \longrightarrow L_{K(1)}K(\mathbb{Z}_p) \longrightarrow \Sigma \text{Fib}(\kappa)$$

that is closer to the arithmetic, and so we work mostly in the $K(1)$ -local setting.

We begin by reviewing notation for some of the basic building blocks of $K(1)$ -local spectra. Let KU_p^\wedge denote p -completed complex periodic K -theory, and let L denote the Adams summand:

$$KU_p^\wedge \simeq L \vee \Sigma^2 L \vee \dots \vee \Sigma^{2p-4} L.$$

Let $J = L_{K(1)}\mathbb{S}$. If we choose an integer l which multiplicatively generates the units of \mathbb{Z}/p^2 , then J is weakly equivalent to the homotopy fiber of $1 - \psi^l : L \rightarrow L$ for the Adams operation ψ^l . In deducing p -complete results from $K(1)$ -local results, we write ku_p^\wedge , ℓ , and j for the connective covers respectively of KU_p^\wedge , L , and J .

The main theorem of Dwyer-Mitchell [19] (as reinterpreted in [6, §2]) splits $L_{K(1)}K(\mathbb{Z})$ canonically as a certain wedge of $K(1)$ -local spectra

$$L_{K(1)}K(\mathbb{Z}) \simeq J \vee Y_0 \vee \cdots \vee Y_{p-2},$$

where Y_i is characterized by the property that $L^*(Y_i)$ is concentrated in degrees congruent to $2i + 1 \bmod 2(p - 1)$ with $L^{2i+1}(Y_i)$ defined as an L^0L -module in terms of a certain abelian Galois group. Because $L^{2i+1}(Y_i)$ is a finitely generated L^0L -module of projective dimension 1, Y_i is the homotopy fiber of a map between wedges of copies of $\Sigma^{2i+1}L$ (which on $L^{2i+1}(-)$ give a projective L^0L -resolution of $L^{2i+1}(Y_i)$). From this it follows that π_*Y_i is concentrated in degrees congruent to $2i$ and $2i + 1 \bmod 2(p - 1)$; moreover, it is free (as a \mathbb{Z}_p -module) in odd degrees. As π_*J is concentrated in degrees congruent to $-1 \equiv 2p - 3 \bmod 2(p - 1)$ and degree 0, any particular homotopy group of $L_{K(1)}K(\mathbb{Z})$ involves only at most a single Y_i and possibly J . The following two results from [6] (q.v. (2.7) and the preceding paragraph) simplify certain arguments.

Proposition 2.3. $Y_{p-2} \simeq *$.

Proposition 2.4. L^1Y_0 is a free L^0L -module of rank 1, and so Y_0 is (non-canonically) weakly equivalent to ΣL .

Let y_i be the connective cover of Y_i for $i \neq 0$ and let y_0 be the 4-connected cover of Y_0 (or equivalently, the $2(p - 1)$ -connected cover); then

$$K(\mathbb{Z})_p^\wedge \simeq j \vee y_0 \vee \cdots \vee y_{p-2}.$$

We have a similar canonical splitting of $L_{K(1)}K(\mathbb{Z}_p^\wedge)$ that takes the form

$$L_{K(1)}K(\mathbb{Z}_p^\wedge) \simeq J \vee \Sigma J' \vee Z_0 \vee \cdots \vee Z_{p-2}$$

where Z_i is non-canonically weakly equivalent to $\Sigma^{2i+1}L$ and has a similar characterization to Y_i ; Z_i was denoted as $\Sigma^{-1}L_{TC}(i + 1)$ in [6]. Here J' is the $K(1)$ -localization of the Moore spectrum $M_{\mathbb{Z}_p^\times}$ for the units of \mathbb{Z}_p , $J' := L_{K(1)}M_{\mathbb{Z}_p^\times}$. Alternatively, J' is canonically weakly equivalent to $(J \wedge^L M_{\mathbb{Z}_p^\times})_p^\wedge$; it is non-canonically weakly equivalent to J since $M_{\mathbb{Z}_p^\times}$ is non-canonically p -equivalent to \mathbb{S} . Let j' be the connective cover of J' , let z_0 be the 4-connected cover of Z_0 , for $0 < i < p - 2$, let z_i be the connective cover of Z_i , and let z_{p-2} be the -2 -connected cover of Z_{p-2} . Then

$$TC(\mathbb{Z})_p^\wedge \simeq j \vee \Sigma j' \vee z_0 \vee \cdots \vee z_{p-2}.$$

The notation of [6] has $z_0 = \Sigma^{-1}\ell_{TC}(p)$, $z_i = \Sigma^{-1}\ell_{TC}(i + 1)$ for $i = 1, \dots, p - 3$, and $z_{p-2} = \Sigma^{-1}\ell_{TC}(0)$. A key result proved in [6, 3.1] is that the cyclotomic trace and the completion map are diagonal on the corresponding pieces.

Theorem 2.5 ([6, 3.1]). *In the notation above, the cyclotomic trace τ decomposes as the wedge of the identity map $j \rightarrow j$ and maps $y_i \rightarrow z_i$ for $i = 0, \dots, p - 2$; the completion map κ decomposes as the wedge of the identity map $J \rightarrow J$ and maps $Y_i \rightarrow Z_i$ for $i = 0, \dots, p - 2$.*

It follows that the cofiber sequences (2.1) and (2.2) decompose into wedges of cofiber sequences. Let X_i and x_i denote the homotopy fiber of the maps $Y_i \rightarrow Z_i$ and $y_i \rightarrow z_i$, respectively. We then have the following consequence of Theorem 2.5.

Corollary 2.6. *The cofiber sequence (2.1) canonically splits as a wedge of the cofiber sequences*

$$\begin{array}{ccccccc}
* & \longrightarrow & j & \xrightarrow{=} & j & \longrightarrow & * \\
j' & \longrightarrow & * & \longrightarrow & \Sigma j' & \xrightarrow{=} & \Sigma j' \\
x_0 & \longrightarrow & y_0 & \longrightarrow & z_0 & \longrightarrow & \Sigma x_0 \\
\vdots & & \vdots & & \vdots & & \vdots \\
x_{p-2} & \longrightarrow & y_{p-2} & \longrightarrow & z_{p-2} & \longrightarrow & \Sigma x_{p-2}.
\end{array}$$

So far we have not used the Main Theorem. Applying the Main Theorem now, we obtain a canonical isomorphism in the stable category

$$\begin{aligned}
\text{Fib}(\kappa) &\simeq \Sigma^{-1} I_{\mathbb{Z}_p}(L_{K(1)}K(\mathbb{Z})) \simeq \Sigma^{-1} I_{\mathbb{Z}_p}(J \vee Y_0 \vee \cdots \vee Y_{p-2}) \\
&\simeq \Sigma^{-1} I_{\mathbb{Z}_p} J \vee \Sigma^{-1} I_{\mathbb{Z}_p} Y_0 \vee \cdots \vee \Sigma^{-1} I_{\mathbb{Z}_p} Y_{p-2}.
\end{aligned}$$

We next identify this wedge decomposition with the wedge decomposition

$$\text{Fib}(\kappa) \simeq J' \vee X_0 \vee \cdots \vee X_{p-2}$$

constructed above.

Theorem 2.7. *The canonical isomorphism in the stable category*

$$\text{Fib}(\kappa) \simeq \Sigma^{-1} I_{\mathbb{Z}_p} J \vee \Sigma^{-1} I_{\mathbb{Z}_p} Y_0 \vee \cdots \vee \Sigma^{-1} I_{\mathbb{Z}_p} Y_{p-2}.$$

identifies J' as $\Sigma^{-1} I_{\mathbb{Z}_p} J$ and X_i as $\Sigma^{-1} I_{\mathbb{Z}_p} Y_{p-2-i}$. Moreover:

- (i) $[X_i, \Sigma^{-1} I_{\mathbb{Z}_p} Y_j] = 0$ unless $i + j \equiv -1 \pmod{p-1}$.
- (ii) $[X_i, \Sigma^{-1} I_{\mathbb{Z}_p} J] = 0$ for all i .
- (iii) $[J', \Sigma^{-1} I_{\mathbb{Z}_p} Y_j] = 0$ for all j .

Proof. To simplify notation, we write D for $\Sigma^{-1} I_{\mathbb{Z}_p}$ inside this proof. The multiplication $L \wedge^L L \rightarrow L$ together with the canonical identification of $\pi_0 L$ as \mathbb{Z}_p induces a map $L \rightarrow I_{\mathbb{Z}_p} L$ that is easily seen to be an isomorphism in the stable category, q.v. [33, 2.6] (this is essentially due to Anderson [1]). This gives us a canonical identification of DL as $\Sigma^{-1} L$, which is the main tool we use.

All statements follow from verification of (i), (ii), and (iii). Recall that Y_i and Z_i both have their L -cohomology $L^*(-)$ concentrated in degrees congruent to $2i + 1 \pmod{2(p-1)}$. Looking at the cofiber sequence

$$\Sigma^{-1} Z_i \longrightarrow X_i \longrightarrow Y_i \longrightarrow Z_i$$

we see that X_i can have L -cohomology only in degrees congruent to $2i$ and $2i + 1 \pmod{2(p-1)}$, or equivalently, that $[X_i, \Sigma^j L] = 0$ unless $j \equiv 2i$ or $j \equiv 2i + 1 \pmod{2(p-1)}$. As discussed above, Y_j fits in a cofiber sequence of the form

$$\bigvee \Sigma^{2j} L \longrightarrow Y_j \longrightarrow \bigvee \Sigma^{2j+1} L \longrightarrow \bigvee \Sigma^{2j+1} L$$

(for some finite wedges of copies of $\Sigma^{2i} L$); it follows that DY_j fits into a cofiber sequence of the form

$$\bigvee \Sigma^{-2j-2} L \longrightarrow \bigvee \Sigma^{-2j-2} L \longrightarrow DY_j \longrightarrow \bigvee \Sigma^{-2j-1} L.$$

Thus, $[X_i, DY_j] = 0$ unless $2i + 2j + 2 \equiv 0 \pmod{2(p-1)}$, or equivalently $i + j + 1 \equiv 0 \pmod{p-1}$. This proves (i).

Writing J as the fiber of a self-map of L , DJ fits then into a cofiber sequence of the form

$$\Sigma^{-1}L \longrightarrow DJ \longrightarrow L \longrightarrow L$$

and it then follows from the work in the previous paragraph that $[X_i, DJ] = 0$ unless $2i \equiv 0$ or $2i + 1 \equiv -1 \pmod{2(p-1)}$. In the first case, $X_0 \simeq *$ by [6, 4.4] which asserts (in the notation here) that $Y_0 \rightarrow Z_0$ is a weak equivalence. In the second case $i = p-2$, $Y_{p-2} \simeq *$ (Proposition 2.3), and so $X_{p-2} \simeq \Sigma^{-1}Z_{p-2}$. But $\Sigma^{-1}Z_{p-2}$ is non-canonically weakly equivalent to $\Sigma^{2p-4}L \simeq \Sigma^{-2}L$, and $[\Sigma^{-2}L, DJ] = 0$. This proves (ii).

Finally, to prove (iii), we note that DY_j is $K(1)$ -local. Since J' is non-canonically weakly equivalent to $J \simeq L_{K(1)}\mathbb{S}$, to see that $[J', DY_j] = 0$ it suffices to see that $\pi_0 DY_j = 0$. Since $\pi_* Y_j$ is concentrated in degrees congruent to $2j$ and $2j + 1 \pmod{2(p-1)}$, we have that $\pi_0 DY_j$ can only possibly be non-zero for $j = p-2$ but as mentioned above, $Y_{p-2} \simeq *$. \square

While we have defined the X_i solely in terms of the fiber sequence, we have defined J' intrinsically, and so the equivalence of J' with $\Sigma^{-1}I_{\mathbb{Z}_p}J$ under the isomorphism in the stable category $\text{Fib}(\kappa) \simeq \Sigma^{-1}I_{\mathbb{Z}_p}K(\mathbb{Z})$ constitutes additional information. In fact, we have a canonical weak equivalence

$$J' \longrightarrow \Sigma^{-1}I_{\mathbb{Z}_p}J$$

that we call the *standard weak equivalence*, constructed as follows. Since J' is defined as the $K(1)$ -localization of the Moore spectrum $M_{\mathbb{Z}_p^\times}$, and $(\mathbb{Z}_p^\times)_p^\wedge$ is a projective \mathbb{Z}_p -module, maps in the stable category from J' into $K(1)$ -local spectra are in canonical one-to-one correspondence with homomorphisms from $(\mathbb{Z}_p^\times)_p^\wedge$ into π_0 . We note that $\Sigma^{-1}I_{\mathbb{Z}_p}J$ is $K(1)$ -local, and to calculate $\pi_0 \Sigma^{-1}I_{\mathbb{Z}_p}J$, we use the fundamental short exact sequence for the \mathbb{Z}_p -Anderson dual: For any spectrum X , there is a canonical natural short exact sequence

$$(2.8) \quad 0 \longrightarrow \text{Ext}(\pi_{-n-1}X, \mathbb{Z}_p) \longrightarrow \pi_n I_{\mathbb{Z}_p}X \longrightarrow \text{Hom}(\pi_{-n}X, \mathbb{Z}_p) \longrightarrow 0.$$

For finitely generated \mathbb{Z}_p -modules, $\text{Hom}(-, \mathbb{Z}_p)$ and $\text{Ext}(-; \mathbb{Z}_p)$ coincide with $\text{Hom}_{\mathbb{Z}_p}(-, \mathbb{Z}_p)$ and $\text{Ext}_{\mathbb{Z}_p}(-; \mathbb{Z}_p)$. In the case of $X = J$, since $\pi_{-2}J = 0$, we then have a canonical identification of $\pi_0 \Sigma^{-1}I_{\mathbb{Z}_p}J$ as $\text{Hom}(\pi_{-1}J, \mathbb{Z}_p)$. The Morava Change of Rings Theorem identifies $\pi_{-1}J$ canonically in terms of continuous group cohomology:

$$\pi_{-1}J \cong H_c^1(\mathbb{Z}_p^\times; \mathbb{Z}_p) \cong \text{Hom}(\mathbb{Z}_p^\times, \mathbb{Z}_p) \cong \text{Hom}((\mathbb{Z}_p^\times)_p^\wedge, \mathbb{Z}_p),$$

q.v. [15, (1.1)], for the continuous action of \mathbb{Z}_p^\times on $\pi_* KU_p^\wedge$ arising from the p -adic interpolation of the Adams operations. This then gives a canonical isomorphism

$$\pi_0 \Sigma^{-1}I_{\mathbb{Z}_p}J \cong \text{Hom}(\text{Hom}((\mathbb{Z}_p^\times)_p^\wedge, \mathbb{Z}_p), \mathbb{Z}_p).$$

Since $(\mathbb{Z}_p^\times)_p^\wedge$ is projective of rank 1, the double dual map is an isomorphism, giving us a canonical isomorphism

$$(\mathbb{Z}_p^\times)_p^\wedge \longrightarrow \pi_0 \Sigma^{-1}I_{\mathbb{Z}_p}J$$

specifying the standard weak equivalence.

On the other hand, we have a canonical map $J' \rightarrow \text{Fib}(\kappa)$ arising from the fiber sequence (2.2) and the $\Sigma J'$ summand of $L_{K(1)}K(\mathbb{Z}_p) \simeq L_{K(1)}K(\mathbb{Q}_p)$. In terms of

maps from $(\mathbb{Z}_p^\times)_p^\wedge$ into $\pi_0 \text{Fib}(\kappa)$, we can therefore identify this map $J' \rightarrow \text{Fib}(\kappa)$ as coming from the canonical identification of the cokernel of

$$\pi_1 L_{K(1)} K(\mathbb{Z}) \longrightarrow \pi_1 L_{K(1)} K(\mathbb{Z}_p)$$

as $(\mathbb{Z}_p^\times)_p^\wedge$ (the p -completion of the cokernel of the map $(\mathbb{Z}[1/p])^\times \rightarrow \mathbb{Q}_p^\times$). The following theorem compares the two maps.

Theorem 2.9. *The composite map $J' \rightarrow \text{Fib}(\kappa) \simeq \Sigma^{-1} I_{\mathbb{Z}_p} K(\mathbb{Z}) \rightarrow \Sigma^{-1} I_{\mathbb{Z}_p} J$ is the standard weak equivalence.*

Because the proof is long and not needed for the rest of the work here, we postpone it to Section 3.

Our final general result is that the fiber sequence defining $\text{Fib}(\kappa)$ is self-dual under local/global duality:

Theorem 2.10. *The following diagram commutes up to the indicated sign*

$$\begin{array}{ccccccc} \text{Fib}(\kappa) & \xrightarrow{\rho} & L_{K(1)} K(\mathbb{Z}) & \xrightarrow{\kappa} & L_{K(1)} K(\mathbb{Z}_p) & \xrightarrow{\partial} & \Sigma \text{Fib}(\kappa) \\ \simeq \downarrow & & \simeq \downarrow & & \simeq \downarrow & (-1) & \simeq \downarrow \\ \Sigma^{-1} I_{\mathbb{Z}_p} (L_{K(1)} K(\mathbb{Z})) & \xrightarrow{\Sigma^{-1} I_{\mathbb{Z}_p} \rho} & \Sigma^{-1} I_{\mathbb{Z}_p} (\text{Fib}(\kappa)) & \xrightarrow{I_{\mathbb{Z}_p} \partial} & I_{\mathbb{Z}_p} L_{K(1)} K(\mathbb{Z}_p) & \xrightarrow{-I_{\mathbb{Z}_p} \kappa} & I_{\mathbb{Z}_p} L_{K(1)} K(\mathbb{Z}) \end{array}$$

where the top sequence is the cofiber sequence (associated to the fiber sequence) defining $\text{Fib}(\kappa)$, the bottom sequence is the \mathbb{Z}_p -Anderson dual of its rotation, and the vertical maps are induced by the K -theoretic Tate-Poitou duality theorem and the K -theoretic local duality theorem.

Proof. The assertion is that $\Sigma\rho$ is \mathbb{Z}_p -Anderson dual to ρ and κ is \mathbb{Z}_p -Anderson dual to ∂ . Given parings

$$\epsilon_i: A_i \wedge^L B_i \longrightarrow I_{\mathbb{Z}_p} \mathbb{S}$$

whose adjoints $\eta_i: B_i \rightarrow F(A_i, I_{\mathbb{S}})$ are weak equivalences, then for maps $f: A_1 \rightarrow A_2$ and $g: B_2 \rightarrow B_1$, $\eta_1 g \circ \eta_2^{-1}$ is \mathbb{Z}_p -Anderson dual to f exactly when the diagram

$$\begin{array}{ccc} A_1 \wedge^L B_2 & \xrightarrow{\text{id} \wedge g} & A_1 \wedge^L B_1 \\ f \wedge \text{id} \downarrow & & \downarrow \epsilon_1 \\ A_2 \wedge^L B_2 & \xrightarrow{\epsilon_2} & I_{\mathbb{Z}_p} \mathbb{S} \end{array}$$

commutes. By construction, the following diagram commutes

$$\begin{array}{ccccc} L_{K(1)} K(\mathbb{Z}[\frac{1}{p}]) \wedge^L L_{K(1)} K(\mathbb{Q}_p) & \xrightarrow{\text{id} \wedge \partial} & L_{K(1)} K(\mathbb{Z}[\frac{1}{p}]) \wedge^L \Sigma \text{Fib}(\kappa) & \xrightarrow{\xi} & \Sigma \text{Fib}(\kappa) \\ \kappa \wedge \text{id} \downarrow & & \nearrow \partial & & \downarrow \Sigma u_{\mathbb{Z}} \\ L_{K(1)} K(\mathbb{Q}_p) \wedge^L L_{K(1)} K(\mathbb{Q}_p) & \xrightarrow{\mu} & L_{K(1)} K(\mathbb{Q}_p) & \xrightarrow{v_{\mathbb{Q}_p}} & I_{\mathbb{Z}_p} \mathbb{S} \end{array}$$

where μ denotes the multiplication, ξ denotes the $L_{K(1)} K(\mathbb{Z}[1/p])$ -module action map, and u and v are the maps described in (1.5) and (1.6), respectively. This

gives the duality between ∂ and κ . To compare ρ and $\Sigma\rho$, consider the diagram

$$\begin{array}{ccc} \mathrm{Fib}(\kappa) \wedge^L \Sigma \mathrm{Fib}(\kappa) & \xrightarrow{\mathrm{id} \wedge \Sigma\rho} & \mathrm{Fib}(\kappa) \wedge^L \Sigma L_{K(1)}K(\mathbb{Z}[\frac{1}{p}]) \\ \rho \wedge \mathrm{id} \downarrow & & \downarrow \\ L_{K(1)}K(\mathbb{Z}[\frac{1}{p}]) \wedge^L \Sigma \mathrm{Fib}(\kappa) & \xrightarrow{\quad} & I_{\mathbb{Z}_p}\mathbb{S} \end{array}$$

where the unlabeled maps are induced by the duality pairing. The down-then-right composite is $u_{\mathbb{Z}}$ composed with the suspension of the non-unital multiplication on $\mathrm{Fib}(\kappa)$, whereas the right-then-down composite is $u_{\mathbb{Z}}$ composed with the suspension of the opposite of the non-unital multiplication on $\mathrm{Fib}(\kappa)$. Since the non-unital multiplication on $\mathrm{Fib}(\kappa)$ is E_{∞} and in particular commutative in the stable category, the diagram commutes. \square

As an immediate consequence, we get duality between the cofiber sequences in Corollary 2.6. Theorem 2.9 indicates the relationship between the j and j' sequences. The relationship between the remaining ones is summarized in the following corollary.

Corollary 2.11. *For each $i = 0, \dots, p-2$ the following diagram commutes up to the indicated sign*

$$\begin{array}{ccccccc} X_i & \xrightarrow{\rho_i} & Y_i & \xrightarrow[\kappa_i]{\kappa} & Z_i & \xrightarrow{\partial_i} & \Sigma X_i \\ \simeq \downarrow & & \simeq \downarrow & & \simeq \downarrow & (-1) & \simeq \downarrow \\ \Sigma^{-1}I_{\mathbb{Z}_p}(Y_{p-2-i}) & \xrightarrow{\Sigma^{-1}I_{\mathbb{Z}_p}\rho_{p-2-i}} & \Sigma^{-1}I_{\mathbb{Z}_p}(X_{p-2-i}) & \xrightarrow{I_{\mathbb{Z}_p}\partial_{p-2-i}} & I_{\mathbb{Z}_p}(Z_{p-2-i}) & \xrightarrow{-I_{\mathbb{Z}_p}\kappa_{p-2-i}} & I_{\mathbb{Z}_p}(Y_{p-2-i}) \end{array}$$

where the top sequence is the cofiber sequence (associated to the fiber sequence) defining X_i , the bottom sequence is the \mathbb{Z}_p -Anderson dual of its rotation, and the vertical maps are induced by the maps

$$\begin{aligned} X_i &\longrightarrow \mathrm{Fib}(\kappa) \longrightarrow \Sigma^{-1}I_{\mathbb{Z}_p}(L_{K(1)}K(\mathbb{Z})) \longrightarrow \Sigma^{-1}I_{\mathbb{Z}_p}(Y_{p-2-i}) \\ Y_i &\longrightarrow L_{K(1)}K(\mathbb{Z}) \longrightarrow \Sigma^{-1}I_{\mathbb{Z}_p}(\mathrm{Fib}(\kappa)) \longrightarrow \Sigma^{-1}I_{\mathbb{Z}_p}(X_{p-2-i}) \\ Z_i &\longrightarrow L_{K(1)}K(\mathbb{Z}_p) \longrightarrow I_{\mathbb{Z}_p}(L_{K(1)}K(\mathbb{Z}_p)) \longrightarrow I_{\mathbb{Z}_p}(Z_{p-2-i}) \end{aligned}$$

arising from local and global K -theoretic duality.

In the case of primes that satisfy the Kummer-Vandiver condition, we can be a bit more specific. A prime p satisfies the *Kummer-Vandiver condition* when p does not divide the order of the class group of $\mathbb{Z}[\zeta_p + \zeta_p^{-1}]$ for $\zeta_p = e^{2\pi i/p}$. In this case, Dwyer-Mitchell [19, 12.2] identifies the homotopy type of the spectra Y_i in terms of the Kubota-Leopoldt p -adic L -function: Given any power series f in the p -adic integers, there is a unique self-map ϕ_f on L in the stable category such that on $\pi_{2(p-1)n}$, ϕ_f is multiplication by $f((1+p)^n - 1)$ (cf. [36, 2.4]). A celebrated theorem of Iwasawa [28] implies in this context that for $i = 1, 3, \dots, p-4$, there exists a self-map of $\Sigma^{2i+1}L$ which on π_{2n+1} is multiplication by the value of the Kubota-Leopoldt p -adic L -function $L_p(-n, \omega^{1+i})$. The spectrum Y_i is non-canonically weakly equivalent to the homotopy fiber of this map. The \mathbb{Z}_p -Anderson self-duality of L then identifies $I_{\mathbb{Z}_p}Y_i$ as (non-canonically) weakly equivalent to the fiber of the self-map of $\Sigma^{-2i}L$ that on π_{2n} is multiplication by $L_p(n, \omega^{1+i})$. In particular, for $i = 2, 4, \dots, p-3$, X_i

is then non-canonically weakly equivalent to the homotopy fiber of the self-map of $\Sigma^{2i-1}L$ that on π_{2n-1} is multiplication by $L_p(n, \omega^{1-i})$. For i even and for j odd, Y_i and X_j are non-canonically weakly equivalent to $\Sigma^{2i+1}L$ and $\Sigma^{2j-1}L$, respectively. (Independently of the Kummer-Vandiver condition $Y_{p-2} \simeq *$ and $X_0 \simeq *$.)

In the case of an odd regular prime, the relevant values of the Kubota-Leopoldt p -adic L -functions are units, and the spectra X_i and x_i are trivial. This is consistent with Rognes' computation [39, 3.3] of the homotopy fiber of the cyclotomic trace as (non-canonically) weakly equivalent to $j \vee \Sigma^{-2}ko_p^\wedge$ in this case. More generally, we have the following corollary.

Corollary 2.12. *Let p be an odd prime that satisfies the Kummer-Vandiver condition. The cofiber sequence*

$$\mathrm{Fib}(\tau) \longrightarrow K(\mathbb{Z})_p^\wedge \longrightarrow TC(\mathbb{Z})_p^\wedge \longrightarrow \Sigma \mathrm{Fib}(\tau)$$

is (non-canonically) weakly equivalent to the wedge of the cofiber sequences

$$\begin{aligned} * &\longrightarrow j \xrightarrow{=} j \longrightarrow * \\ j' &\longrightarrow * \longrightarrow \Sigma j' \xrightarrow{=} \Sigma j' \\ x_i &\longrightarrow \Sigma(\Sigma^{2i}\ell) \xrightarrow{\Sigma\lambda_i^e} \Sigma(\Sigma^{2i}\ell) \longrightarrow \Sigma x_i \quad i = 2, 4, \dots, p-3 \\ \Sigma^{2i}\ell &\longrightarrow y_i \longrightarrow \Sigma(\Sigma^{2i}\ell) \xrightarrow{\Sigma\lambda_i^o} \Sigma(\Sigma^{2i}\ell) \quad i = 1, 3, \dots, p-4 \\ * &\longrightarrow \Sigma^{2p-1}\ell \xrightarrow{=} \Sigma^{2p-1}\ell \longrightarrow * \quad (i = 0) \\ \Sigma^{-2}\ell &\longrightarrow * \longrightarrow \Sigma^{-1}\ell \xrightarrow{=} \Sigma^{-1}\ell \quad (i = p-2) \end{aligned}$$

where λ_i^e is the unique self-map of $\Sigma^{2i}\ell$ that on π_{2n} is multiplication by the value $L_p(n+1, \omega^{-i})$, λ_i^o is the unique self-map of $\Sigma^{2i}\ell$ that on π_{2n} is multiplication by the value $L_p(-n, \omega^{1+i})$, and L_p denotes the Kubota-Leopoldt p -adic L -function.

3. PROOF OF THEOREM 2.9

This section is devoted to the proof of Theorem 2.9. None of the details of the argument are needed anywhere else in the paper. As discussed above the statement of Theorem 2.9, maps from J' to $\mathrm{Fib}(\kappa)$ are determined by maps from $(\mathbb{Z}_p^\times)_p^\wedge$ to $\pi_0 \mathrm{Fib}(\kappa)$; we have two isomorphisms of $(\mathbb{Z}_p^\times)_p^\wedge$ with $\pi_0 \mathrm{Fib}(\kappa)$ and we need to show that they are the same. It is slightly easier to work with the \mathbb{Z}_p -duals instead. We can canonically identify the \mathbb{Z}_p -dual of $\pi_0 \mathrm{Fib}(\kappa)$ as $\pi_{-1}L_{K(1)}K(\mathbb{Z})$: the fundamental short exact sequence (2.8) for $\pi_0\Sigma^{-1}I_{\mathbb{Z}_p}K(\mathbb{Z})$ gives an isomorphism

$$\pi_0\Sigma^{-1}I_{\mathbb{Z}_p}K(\mathbb{Z}) \cong \mathrm{Hom}(\pi_{-1}L_{K(1)}K(\mathbb{Z}), \mathbb{Z}_p)$$

since $\pi_{-2}L_{K(1)}K(\mathbb{Z}) = 0$ (as $Y_{p-2} \simeq *$). The identification of $\pi_{-1}J$ as $\mathrm{Hom}(\mathbb{Z}_p^\times, \mathbb{Z}_p)$ above and the canonical map $J \rightarrow L_{K(1)}K(\mathbb{Z})$ gives one isomorphism of $\pi_{-1}L_{K(1)}K(\mathbb{Z})$ with $\mathrm{Hom}(\mathbb{Z}_p^\times, \mathbb{Z}_p)$, which we denote as α . The isomorphism of $\pi_0 \mathrm{Fib}(\tau)$ with $(\mathbb{Z}_p^\times)_p^\wedge$ as the quotient of $\pi_1 L_{K(1)}K(\mathbb{Q}_p) \cong (\mathbb{Q}_p^\times)_p^\wedge$ gives another isomorphism of $\pi_{-1}L_{K(1)}K(\mathbb{Z})$ with $\mathrm{Hom}(\mathbb{Z}_p^\times, \mathbb{Z}_p)$ which we denote as η . We need to prove that $\eta = \alpha$.

We have an intrinsic identification of $\pi_{-1}L_{K(1)}K(\mathbb{Z})$ coming from Thomason's descent spectral sequence, which canonically identifies $\pi_{-1}L_{K(1)}K(\mathbb{Z}) \cong \pi_{-1}L_{K(1)}K(\mathbb{Z}[1/p])$

as $H_{\text{ét}}^1(\text{Spec } \mathbb{Z}[1/p]; \mathbb{Z}_p)$ (continuous étale cohomology). Let \mathbb{Q}_S denote the maximal algebraic extension of \mathbb{Q} that is unramified except over $S = \{p\}$, and let $G_S = \text{Gal}(\mathbb{Q}_S/\mathbb{Q})$. The abelianization G_S^{ab} of G_S corresponds to the maximal abelian extension of \mathbb{Q} that is unramified except over p , which is $\mathbb{Q}(\mu_{p^\infty})$ (where μ_{p^∞} denotes the group of p^n th roots of unity for all n). We have the standard identification of the continuous étale cohomology $H_{\text{ét}}^1(\text{Spec } \mathbb{Z}[1/p]; \mathbb{Z}_p)$ as the Galois cohomology $H_{\text{Gal}}^1(\mathbb{Q}_S/\mathbb{Q}; \mathbb{Z}_p)$ [34, II.2.9], which we can identify as the abelian group of continuous homomorphisms

$$\text{Hom}_c(G_S, \mathbb{Z}_p) \cong \text{Hom}_c(G_S^{\text{ab}}, \mathbb{Z}_p).$$

We have a further isomorphism

$$G_S^{\text{ab}} = \text{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}) \cong \varinjlim_n \text{Gal}(\mathbb{Q}(\mu_{p^n})/\mathbb{Q}) \cong \varinjlim_n (\mathbb{Z}/p^n)^\times \cong \mathbb{Z}_p^\times,$$

where the first isomorphism is inverse to the isomorphism $(\mathbb{Z}/p^n)^\times \cong \text{Gal}(\mathbb{Q}(\mu_{p^n})/\mathbb{Q})$ taking an element $u \in (\mathbb{Z}/p^n)^\times$ to the automorphism of $\mathbb{Q}(\mu_{p^n})$ induced by the automorphism $\zeta \mapsto \zeta^u$ on μ_{p^n} . This then constructs an isomorphism

$$\gamma: \pi_{-1}L_{K(1)}K(\mathbb{Z}) \longrightarrow \text{Hom}_c(\mathbb{Z}_p^\times, \mathbb{Z}_p) \cong \text{Hom}(\mathbb{Z}_p^\times, \mathbb{Z}_p).$$

First we show $\gamma = \alpha$. Choose a prime number l such that l is a topological generator of \mathbb{Z}_p^\times , or equivalently, a generator of \mathbb{Z}/p^2 , and consider the quotient map $\mathbb{Z}[1/p] \rightarrow \mathbb{Z}/l = \mathbb{F}_l$. By celebrated work of Quillen, the composite map

$$j \longrightarrow K(\mathbb{Z})_p^\wedge \longrightarrow K(\mathbb{F}_l)_p^\wedge$$

is a weak equivalence and an embedding of \mathbb{F}_l^\times in \mathbb{C}^\times induces a weak equivalence $K(\mathbb{F}_l)_p^\wedge \rightarrow ku_p^\wedge$ with the automorphism Φ on $K(\mathbb{F}_l)_p^\wedge$ induced by the Frobenius automorphism of \mathbb{F}_l mapping to the Adams operation Ψ^l on ku_p^\wedge (independently of the choice of embedding). We will also write Φ for the corresponding automorphism of $L_{K(1)}K(\mathbb{F}_l)$. For any functorial model of $L_{K(1)}K(-)$, the induced map from $L_{K(1)}K(\mathbb{F}_l)$ into the homotopy fixed points of Φ (the homotopy equalizer of Φ and the identity on $L_{K(1)}K(\mathbb{F}_l)$) is a weak equivalence. Writing $L_{K(1)}K(\mathbb{F}_l)^{h\Phi}$ for the homotopy fixed points of Φ , the map $L_{K(1)}K(\mathbb{F}_l) \rightarrow L_{K(1)}K(\mathbb{F}_l)^{h\Phi}$ is the unique map that takes the unit element of $\pi_0(L_{K(1)}K(\mathbb{F}_l))$ to the unique element of $\pi_0(L_{K(1)}K(\mathbb{F}_l)^{h\Phi})$ that maps to the unit element of $\pi_0(L_{K(1)}K(\mathbb{F}_l))$. This gives a canonical identification of $\pi_{-1}(L_{K(1)}K(\mathbb{F}_l))$ as $H^1(\langle \Phi \rangle; \mathbb{Z}_p)$, where $\langle \Phi \rangle$ denotes the cyclic group generated by Φ ; we have used the canonical isomorphism $\pi_0(L_{K(1)}K(\mathbb{F}_l)) \cong \mathbb{Z}_p$ induced by the unit and we note that this isomorphism is consistent with the canonical isomorphism $\pi_0 KU_p^\wedge \cong \mathbb{Z}_p$ under the weak equivalence $L_{K(1)}K(\mathbb{F}_l) \rightarrow KU_p^\wedge$ (independently of the choice of the embedding $\mathbb{F}_l^\times \rightarrow \mathbb{C}^\times$). Under the identification of $\pi_{-1}J$ as $H_c^1(\mathbb{Z}_p^\times; \mathbb{Z}_p)$ above, the composite map

$$J \longrightarrow L_{K(1)}K(\mathbb{Z}) \longrightarrow L_{K(1)}K(\mathbb{F}_l)$$

induces on π_{-1} the map

$$H_c^1(\mathbb{Z}_p^\times; \mathbb{Z}_p) \longrightarrow H^1(\langle \Phi \rangle; \mathbb{Z}_p)$$

induced by the inclusion of l in \mathbb{Z}_p^\times (the inclusion of Ψ^l in the group of p -adically interpolated Adams operations). This gives us information about α . In terms of the

identification of $\pi_{-1}(L_{K(1)}K(\mathbb{F}_l))$ as $H_{\text{ét}}^1(\mathbb{F}_l; \mathbb{Z}_p)$ from Thomason's descent spectral sequence, the map

$$H_{\text{ét}}^1(\mathbb{F}_l; \mathbb{Z}_p) \cong H_{\text{Gal}}^1(\bar{\mathbb{F}}_l/\mathbb{F}_l; \mathbb{Z}_p) \longrightarrow H^1(\langle \Phi \rangle; \mathbb{Z}_p)$$

is induced by the inclusion of the Frobenius in $\text{Gal}(\bar{\mathbb{F}}_l/\mathbb{F}_l)$. By naturality, the composite map

$$H_{\text{ét}}^1(\mathbb{Z}[1/p]; \mathbb{Z}_p) \cong H_c^1(G_S; \mathbb{Z}_p) \longrightarrow H_c^1(\text{Gal}(\bar{\mathbb{F}}_l; \mathbb{Z}_p)) \longrightarrow H^1(\langle \Phi \rangle; \mathbb{Z}_p)$$

is induced by the inclusion of Φ in G_S as the automorphism of (the p -integers in) \mathbb{Q}_S that does the automorphism $\zeta \mapsto \zeta^l$ on μ_{p^∞} . This then shows that $\gamma = \alpha$.

We now compare γ and η . Here it is easiest to work first in terms of $L_{K(1)}K(\mathbb{Q}_p)$. Using the standard identification of $\pi_1 L_{K(1)}K(\mathbb{Q}_p)$ as the p -completion of the units, we have a \mathbb{Q}_p -analogue of η using local duality: Let

$$\eta_p: \pi_{-1} L_{K(1)}K(\mathbb{Q}_p) \longrightarrow \text{Hom}((\mathbb{Q}_p^\times)_p^\wedge, \mathbb{Z}_p) \cong \text{Hom}(\mathbb{Q}_p^\times, \mathbb{Z}_p)$$

be the isomorphism derived from the isomorphism $\pi_1 L_{K(1)}K(\mathbb{Q}_p) \cong (\mathbb{Q}_p^\times)_p^\wedge$ by Anderson duality. We then have a commutative diagram

$$\begin{array}{ccccc} \pi_{-1} L_{K(1)}K(\mathbb{Z}) & \xrightarrow{\cong} & \pi_{-1} L_{K(1)}K(\mathbb{Z}[1/p]) & \longrightarrow & \pi_{-1} L_{K(1)}K(\mathbb{Q}_p) \\ \eta \downarrow \cong & & & & \cong \downarrow \eta_p \\ \text{Hom}(\mathbb{Z}_p^\times, \mathbb{Z}_p) & \longrightarrow & & \longrightarrow & \text{Hom}(\mathbb{Q}_p^\times, \mathbb{Z}_p) \end{array}$$

by the compatibility of local and global duality. To produce a local analogue of γ , we use the Artin symbol θ in local class field theory [41, §3.1]. The Artin symbol gives an isomorphism between the finite completion of the units of \mathbb{Q}_p and the Galois group of the maximal abelian extension $\bar{\mathbb{Q}}_p^{\text{ab}}$ of \mathbb{Q}_p : For $x \in \mathbb{Q}_p^\times$, writing $x = ap^m$ for $a \in \mathbb{Z}_p^\times$, the Artin symbol takes x to the unique element $\theta(x)$ of $\text{Gal}(\bar{\mathbb{Q}}_p^{\text{ab}}/\mathbb{Q}_p)$ that acts on the p^n th roots of unity μ_{p^n} by raising to the $1/a$ power and acts on the maximal unramified extension $(\mathbb{Q}_p)^{\text{nr}}$ of \mathbb{Q}_p by the m th power of a lift of the Frobenius. Using the isomorphism

$$\pi_{-1} L_{K(1)}K(\mathbb{Q}_p) \cong H_{\text{ét}}^1(\mathbb{Q}_p; \mathbb{Z}_p)$$

from Thomason's descent spectral sequence and the canonical isomorphism

$$H_{\text{ét}}^1(\mathbb{Q}_p; \mathbb{Z}_p) \cong H_{\text{Gal}}^1(\bar{\mathbb{Q}}_p/\mathbb{Q}_p; \mathbb{Z}_p) \cong \text{Hom}_c(\text{Gal}(\bar{\mathbb{Q}}_p^{\text{ab}}/\mathbb{Q}_p), \mathbb{Z}_p)$$

(as above), the Artin symbol induces an isomorphism

$$-\gamma_p: \pi_{-1} L_{K(1)}K(\mathbb{Q}_p) \longrightarrow \text{Hom}(\mathbb{Q}_p^\times, \mathbb{Z}_p).$$

We have implicitly defined an isomorphism γ_p : The formula for the Artin symbol implies that the following diagram commutes

$$\begin{array}{ccccc} \pi_{-1} L_{K(1)}K(\mathbb{Z}) & \xrightarrow{\cong} & \pi_{-1} L_{K(1)}K(\mathbb{Z}[1/p]) & \longrightarrow & \pi_{-1} L_{K(1)}K(\mathbb{Q}_p) \\ \gamma \downarrow \cong & & & & \cong \downarrow \gamma_p \\ \text{Hom}(\mathbb{Z}_p^\times, \mathbb{Z}_p) & \longrightarrow & & \longrightarrow & \text{Hom}(\mathbb{Q}_p^\times, \mathbb{Z}_p) \\ = \downarrow & & & & \cong \downarrow \\ \text{Hom}(\mathbb{Z}_p^\times, \mathbb{Z}_p) & \xrightarrow{\quad (\text{id}, 0) \quad} & & \longrightarrow & \text{Hom}(\mathbb{Z}_p^\times, \mathbb{Z}_p) \times \mathbb{Z}_p^\wedge \end{array}$$

(where the bottom right vertical isomorphism is induced by the the ap^m decomposition of \mathbb{Q}_p^\times as $\mathbb{Z}_p^\times \times \mathbb{Z}$). In other words, omitting notation for the isomorphism arising from Thomason's descent spectral sequence and the usual isomorphism $H_{\text{ét}}^1(\mathbb{Q}_p; \mathbb{Z}_p) \cong \text{Hom}(\text{Gal}(\bar{\mathbb{Q}}_p^{\text{ab}}/\mathbb{Q}_p), \mathbb{Z}_p)$, γ_p is the \mathbb{Z}_p -dual of $-\theta$. The Artin symbol has a cohomological characterization [41, §2.3, Prop. 1]: For a character $\rho: \text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \rightarrow \mathbb{Q}/\mathbb{Z}$ and $x \in \mathbb{Q}_p^\times$,

$$\rho(\theta(x)) = -\text{inv}(x \cup \rho)$$

(cf. [37, p. 386]) where on the right we interpret x as an element of $H_{\text{ét}}^1(\mathbb{Q}_p; \mathbb{Z}_p)$ and ρ as an element of $H_{\text{ét}}^1(\mathbb{Q}_p; \mathbb{Q}/\mathbb{Z})$, while the symbol \cup denotes the cup product on étale cohomology

$$H_{\text{ét}}^1(\mathbb{Q}_p; \mathbb{Z}_p(1)) \otimes H_{\text{ét}}^1(\mathbb{Q}_p; \mathbb{Q}/\mathbb{Z}) \longrightarrow H_{\text{ét}}^2(\mathbb{Q}; \mathbb{Q}/\mathbb{Z}(1)),$$

and inv denotes the map induced by the Hasse invariant as in Section 1. Because

$$\text{inv}(x \cup y): H_{\text{ét}}^1(\mathbb{Q}_p; \mathbb{Z}_p(1)) \otimes H_{\text{ét}}^1(\mathbb{Q}_p; \mathbb{Q}_p/\mathbb{Z}_p) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

is the local duality pairing, restricting to

$$H_{\text{ét}}^1(\mathbb{Q}_p; \mathbb{Z}/p^n(1)) \otimes H_{\text{ét}}^1(\mathbb{Q}_p; \mathbb{Z}/p^n) \longrightarrow \mathbb{Z}/p^n,$$

taking the inverse limit, and applying the isomorphism from Thomason's descent spectral sequence gives us the Anderson duality pairing

$$\pi_1 L_{K(1)} K(\mathbb{Q}_p) \otimes \pi_{-1} L_{K(1)} K(\mathbb{Q}_p) \longrightarrow \mathbb{Z}_p.$$

We conclude that $\eta_p = \gamma_p$. Since the map $\pi_{-1} L_{K(1)} K(\mathbb{Z}) \rightarrow \pi_{-1} L_{K(1)} K(\mathbb{Q}_p)$ is an injection, we conclude that $\eta = \gamma$.

4. COMPLETED COHOMOLOGY AND THE FIBER OF THE CYCLOTOMIC TRACE

The ideas and proposals of Breuil [10] have led to a great deal of new activity on a p -adic version of the Langlands correspondence. A key ingredient in this theory is a suitable notion of the space of p -adic automorphic forms. There are various ways to approach the construction of such a space. In order to have a definition which is “representation-theoretic”, Calegari and Emerton [12, 21] have proposed that the p -adically completed (co)homology of congruence subgroups should play the role of the space of p -adic automorphic forms. As a consequence, computing these completed (co)homology groups and more generally understanding their structure is a problem of fundamental importance.

Calegari and Emerton [13] showed that, in contrast to the uncompleted (co)homology of congruence subgroups, the completed versions stabilize (in a sense we review below). In a subsequent paper, Calegari [11] proved a number of quantitative results about the completed cohomology groups. In the process, he described the relationship of the completed cohomology of congruence subgroups to algebraic K -theory. As we explain in this section, Calegari's work can be viewed as giving an interpretation of the completed (co)homology in terms of the fiber of the cyclotomic trace. In light of this connection, our main theorems imply a specific conjecture of Calegari and serve both to provide a spectrum-level lift of the completed homology groups and also to sharpen some of Calegari's low-degree calculations.

We begin by reviewing the definition of the completed cohomology groups, following the notation of Calegari and Emerton. Fix a number field F and let Γ_N denote $\text{SL}_N(\mathcal{O}_F)$. For an integer M , the principal congruence subgroup of level M ,

$\Gamma_N(M)$, is the kernel of the reduction map $\Gamma_N \rightarrow \mathrm{SL}_N(\mathcal{O}_F/M)$. For a prime p , we define the completed homology and cohomology groups of Γ_N to be

$$\begin{aligned}\tilde{H}_*(\mathrm{SL}_N, F_p) &= \lim_r H_*(\Gamma_N(p^r), F_p) \\ \tilde{H}^*(\mathrm{SL}_N, F_p) &= \operatorname{colim}_r H^*(\Gamma_N(p^r), F_p)\end{aligned}$$

and

$$\begin{aligned}\tilde{H}_*(\mathrm{SL}_N, \mathbb{Z}_p) &= \lim_r H_*(\Gamma_N(p^r), \mathbb{Z}_p) \\ \tilde{H}^*(\mathrm{SL}_N, \mathbb{Q}_p/\mathbb{Z}_p) &= \operatorname{colim}_r H^*(\Gamma_N(p^r), \mathbb{Q}_p/\mathbb{Z}_p).\end{aligned}$$

The main result of Calegari-Emerton [13] is that for each d the homology groups $\tilde{H}_d(\mathrm{SL}_N, \mathbb{Z}_p)$ stabilize as $N \rightarrow \infty$; denote the associated colimit by $\tilde{H}_d(\mathrm{SL}, \mathbb{Z}_p)$.

One of the main goals of Calegari's subsequent paper [11] is to provide an interpretation of the completed homology group $\tilde{H}_*(\mathrm{SL}, \mathbb{Z}_p)$ in terms of algebraic K -theory. Calegari defines the completed K -theory groups $\tilde{K}_*(\mathcal{O}_F)$ at a prime p to be the homotopy groups of the p -completion of the fiber of the completion map $K(\mathcal{O}_F) \rightarrow K((\mathcal{O}_F)_p^\wedge)$ of K -theory spaces [11, 1.1]. Furthermore, he constructs an isomorphism

$$\tilde{H}_*(\mathrm{SL}, \mathbb{Z}_p) \cong \lim_n H_*(Y_\infty, \mathbb{Z}/p^n),$$

where Y_∞ is a space closely related to the fiber of the completion map on algebraic K -theory spaces [11, 1.17]. As a consequence, the Hurewicz theorem can be used to obtain information about the completed homology groups from computations of the completed K -theory groups.

Based on the Tate-Poitou sequence, Calegari [11, 1.5] conjectures isomorphisms

$$\tilde{K}_{2n-1}(\mathcal{O}_F) \cong (H_{\text{ét}}^0(\mathcal{O}_F[1/p]; \mathbb{Q}_p/\mathbb{Z}_p(-n)) \oplus H_{\text{ét}}^2(\mathcal{O}_F[1/p]; \mathbb{Q}_p/\mathbb{Z}_p(1-n)))^*$$

(where $*$ denotes Pontryagin dual) and further conjectures that the groups $\tilde{K}_n(\mathcal{O}_F)$ fit into a long exact sequence with $K_n(\mathcal{O}_F; \mathbb{Z}_p)$ and $K_n((\mathcal{O}_F)_p^\wedge; \mathbb{Z}_p)$. Our K -theoretic Tate-Poitou duality theorem precisely affirms this conjecture, yielding the construction of a spectrum with homotopy groups $\tilde{K}_*(\mathcal{O}_F)$ and continuous spectrum homology given by $\tilde{H}_*(\mathrm{SL}, \mathbb{Z}_p)$ [11, 1.19]. Moreover, the affirmed conjecture sharpens [11, 0.2] (an explicit calculation of completed homology) by making the conclusion unconditional.

5. PRELIMINARIES

The proof of the K -theoretic Tate-Poitou duality theorem depends on an interpretation of the homotopy fiber $\mathrm{Fib}(\kappa)$ in the statement as “extension by zero” in the context of presheaves of symmetric spectra for a presheaf arising from $K(1)$ -localized algebraic K -theory. It also depends on the module pairing arising from the ring structure on K -theory. The purpose of this section is to review relevant aspects of the general homotopy theory of presheaves of symmetric spectra and the specific properties of the algebraic K -theory presheaf, assembling results from Thomason [43], Jardine [32, 30, 31], and Dugger-Hollander-Isaksen [18], keeping track of which constructions preserve the pairing.

Throughout the paper, we fix a hierarchy of Grothendieck universes $\mathcal{U}_0 \subsetneq \mathcal{U}_1 \subsetneq \mathcal{U}_2$. A “scheme” will be understood as a scheme in \mathcal{U}_0 -sets, and for a scheme A , we denote by $A_{\text{ét}}$ the (small) étale site, which we understand as a category

with a \mathcal{U}_1 -set of objects (and \mathcal{U}_0 morphism sets). Its objects are the étale maps $V \rightarrow A$ and its covers are the surjective \mathcal{U}_0 -families $\{V_\alpha \rightarrow V\}$. We understand a “symmetric spectrum” to be a symmetric spectrum in \mathcal{U}_1 -sets; the category of symmetric spectra is then \mathcal{U}_1 -complete and \mathcal{U}_1 -cocomplete and in particular $A_{\text{ét}}$ is “small” from the perspective of this category. The category of presheaves of symmetric spectra on $A_{\text{ét}}$ is the category $\mathcal{P}(A_{\text{ét}}, \mathcal{S})$ of contravariant functors from $A_{\text{ét}}$ to symmetric spectra. With the usual tricks, it would be possible to model symmetric spectra and schemes on the same universe, but this setup avoids having to do any further cardinal bounds or finality/cofinality arguments.

In this section, we fix a scheme A and generally understand presheaves to be presheaves of symmetric spectra on the site $A_{\text{ét}}$. Most of the exposition applies to more general sites with some modifications or additional hypotheses.

5.1. The homotopy theory of presheaves of symmetric spectra. This subsection reviews the basics of the homotopy theory of presheaves of symmetric spectra on $A_{\text{ét}}$. We work with symmetric spectra rather than (pre)spectra as the category of presheaves of symmetric spectra inherits a symmetric monoidal structure from the category of symmetric spectra: For presheaves of symmetric spectra \mathcal{F} and \mathcal{G} , we obtain a presheaf of symmetric spectra $\mathcal{F} \wedge \mathcal{G}$ defined by

$$(\mathcal{F} \wedge \mathcal{G})(V) = \mathcal{F}(V) \wedge \mathcal{G}(V).$$

The unit for this smash product is the constant presheaf with value \mathbb{S} , the sphere spectrum.

A disadvantage of working with symmetric spectra is the confusion of what is meant by “homotopy groups”, as the underlying point-set spectrum of a symmetric spectrum is not necessarily weakly equivalent to the object of the stable category that the symmetric spectrum represents. Whenever we refer to the homotopy groups of a symmetric spectrum, we will mean the right derived homotopy groups, which are the homotopy groups of the object the symmetric spectrum represents in the stable category, or equivalently, the homotopy groups of a fibrant replacement (cf. Convention in [5, p. 1060]). For a presheaf of symmetric spectra on $A_{\text{ét}}$, applying the homotopy groups functor at each object of $A_{\text{ét}}$, we get a presheaf of graded abelian groups, which we then sheafify to obtain a sheaf of graded abelian groups.

Definition 5.1.1. Given a presheaf \mathcal{F} of symmetric spectra, the *sheaf of homotopy groups* $\tilde{\pi}_*\mathcal{F}$ is the sheaf of graded abelian groups obtained as the sheafification of the presheaf of (right derived) homotopy groups. A map $\mathcal{F} \rightarrow \mathcal{G}$ of presheaves of symmetric spectra is a *weak equivalence* if it induces an isomorphism $\tilde{\pi}_*\mathcal{F} \rightarrow \tilde{\pi}_*\mathcal{G}$.

Since $A_{\text{ét}}$ has enough points, we can reformulate the notion of weak equivalence in terms of stalks. A point in $A_{\text{ét}}$ corresponds to a point on A , or better, its corresponding geometric point.

Definition 5.1.2. For $x: \text{spec } k(x) \rightarrow A$ a point of A , choose a model $k(x)^s$ for the separable closure of $k(x)$, and write \bar{x} for the composite map $\text{spec } k(x)^s \rightarrow \text{spec } k(x) \rightarrow A$. For a presheaf of symmetric spectra \mathcal{F} , the stalk $\mathcal{F}_{\bar{x}}$ of \mathcal{F} at \bar{x} is defined to be

$$\mathcal{F}_{\bar{x}} = \text{colim } \mathcal{F}(W)$$

where the colimit is taken over the category of commutative diagrams

$$\begin{array}{ccc} \mathrm{spec} k(x)^s & \longrightarrow & W \\ \downarrow & & \downarrow \\ \mathrm{spec} k(x) & \longrightarrow & A \end{array}$$

with $W \rightarrow A$ étale.

The category above is projectively filtered, the colimit is inductively filtered, and the (right derived) homotopy groups of the stalk $\pi_* \mathcal{F}_{\bar{x}}$ are canonically isomorphic to the stalk of the homotopy group sheaf $(\tilde{\pi}_* \mathcal{F})_{\bar{x}}$. Therefore, a map of presheaves of symmetric spectra on $A_{\text{ét}}$ is a weak equivalence if and only if it is a weak equivalence on all stalks.

Proposition 5.1.3. *A map $\mathcal{F} \rightarrow \mathcal{G}$ of presheaves of symmetric spectra on $A_{\text{ét}}$ is a weak equivalence if and only if it induces a weak equivalence $\mathcal{F}_{\bar{x}} \rightarrow \mathcal{G}_{\bar{x}}$ on each stalk.*

Jardine [31] constructs a model structure on presheaves of symmetric spectra on a site \mathcal{C} with weak equivalences *a fortiori* the weak equivalences of Definition 5.1.1 (see Theorem 13 of *ibid.* and [30, 2.6ff]). We call the fibrant objects in this model structure the *Jardine fibrant* presheaves of symmetric spectra. As the model structure admits functorial factorization, there is a Jardine fibrant replacement functor. Implicitly, this establishes the homotopy theory we need and the Jardine fibrant presheaves (and Jardine fibrant replacement) provide the construction of many of the right derived functors we require. We will say more about Jardine fibrant presheaves below in the discussion of hypercover descent conditions.

The Jardine fibrant replacement functor is not known to be symmetric monoidal (but see Proposition 5.3.2 below); for this reason, it is convenient to have weaker fibrancy notions as well that apply to the constituent symmetric spectra or the constituent simplicial sets. We consistently use the following terminology.

Terminology 5.1.4. For a presheaf of symmetric spectra \mathcal{F} or a map of presheaves of symmetric spectra $f: \mathcal{F} \rightarrow \mathcal{F}'$, we say that a property of symmetric spectra or maps of symmetric spectra holds *objectwise* if it holds for each $\mathcal{F}(V)$ or $f(V): \mathcal{F}(V) \rightarrow \mathcal{F}'(V)$ for every object V in $A_{\text{ét}}$. For a symmetric spectrum T or map of symmetric spectra $g: T \rightarrow T'$, we say that a property of simplicial sets or maps of simplicial sets holds *levelwise* if it holds for each T_n or $T_n \rightarrow T'_n$ for each natural number n .

In particular, we say that a presheaf of symmetric spectra \mathcal{F} is *objectwise fibrant* if each symmetric spectrum $\mathcal{F}(V)$ is fibrant in the stable model category of symmetric spectra (i.e., is a levelwise fibrant Ω -spectrum). Likewise, a map $\mathcal{F} \rightarrow \mathcal{F}'$ is an objectwise weak equivalence if it is a weak equivalence $\mathcal{F}(V) \rightarrow \mathcal{F}'(V)$ for all V ; an objectwise weak equivalence is in particular a weak equivalence. A fibrant replacement functor for symmetric spectra then gives us an objectwise fibrant replacement functor for presheaves of symmetric spectra. It will often suffice to work with an even more mild point-set condition: \mathcal{F} is objectwise levelwise fibrant if for each V and each natural number n , the simplicial set $\mathcal{F}(V)_n$ is fibrant (i.e., a Kan complex). A levelwise fibrant replacement functor for symmetric spectra then gives us an objectwise levelwise replacement functor for presheaves of symmetric spectra. An example of a levelwise fibrant replacement functor for symmetric spectra is the

levelwise applied composite functor $S|-|$, where $|-|$ denotes geometric realization and S its right adjoint (the singular simplicial set functor); this functor also has the advantage of being lax symmetric monoidal for the smash product of symmetric spectra.

5.2. Hypercover descent. The homotopical replacement for the sheaf condition in the context of presheaves of symmetric spectra is a homotopical descent condition called “hypercovder descent”. Hypercovers are a generalization of Čech covers and we give a quick review of the definition for the étale site; for more details see [3, Chapter 8], [2, II.v.7], or [18, §4, §10]. For $n \geq 0$ and V an object in $A_{\text{ét}}$, let $s_{\leq n}A_{\text{ét}}/V$ denote the category of truncated simplicial objects in $A_{\text{ét}}/V$ (of dimension $\leq n$). There is an evident n -skeleton truncation functor $\text{sk}_n: sA_{\text{ét}}/V \rightarrow s_{\leq n}A_{\text{ét}}/V$, which has a right adjoint $\text{cosk}_n: s_{\leq n}A_{\text{ét}}/V \rightarrow sA_{\text{ét}}/V$ since $A_{\text{ét}}/V$ admits fiber products. A hypercover of V is then an object U_{\bullet} of $sA_{\text{ét}}/V$ such that the structure map $U_0 \rightarrow V$ is a cover and for all $n > 0$, the map $U_n \rightarrow (\text{cosk}_{n-1} U_{\bullet})_n$ is a cover. A hypercover is a generalization of the notion of a Čech cover: Given a cover $V_{\alpha} \rightarrow V$ and setting $U_0 = \coprod V_{\alpha}$, requiring that $U_n = (\text{cosk}_{n-1} U_{\bullet})_n$ precisely recovers the usual simplicial Čech complex associated to the cover $U_0 \rightarrow V$.

Dugger-Hollander-Isaksen [18] introduces and studies a condition on presheaves of simplicial sets called “hypercovder descent” [18, 4.3], which has an obvious analogue for presheaves of symmetric spectra. (As the hypercovers defined above correspond to “internal hypercovers” in the Dugger-Hollander-Isaksen terminology [18, §10], we follow the implicit definition in [18, 10.2].)

Definition 5.2.1. A presheaf \mathcal{F} of symmetric spectra on $A_{\text{ét}}$ satisfies *hypercovder descent* means that for any object $V \in A_{\text{ét}}$ and hypercover $U_{\bullet} \rightarrow V$, the natural map

$$\mathcal{F}(V) \longrightarrow \text{holim}_{\Delta^{\text{op}}} \mathcal{F}(U_{\bullet})$$

is a weak equivalence and for any \mathcal{U}_0 -indexed family of objects $\{V_{\alpha}\}$ in $A_{\text{ét}}$, the natural map

$$\mathcal{F}(\coprod V_{\alpha}) \longrightarrow R \coprod \mathcal{F}(V_{\alpha})$$

is a weak equivalence.

Here $R \coprod$ denotes the right derived functor of product; it can be formed as the product applied objectwise levelwise to an objectwise fibrant replacement of \mathcal{F} . Likewise, holim here denotes the right derived functor of limit; an explicit construction of the homotopy limit may be given as the Bousfield-Kan homotopy limit construction applied to an objectwise fibrant replacement of \mathcal{F} . The derived version of the product and homotopy limit (by definition) have the property that they are invariant in objectwise equivalences of presheaves of symmetric spectra. Conversely, [18, 1.3] implies the following proposition, which gives hypercovder descent its power.

Proposition 5.2.2. *For presheaves of symmetric spectra satisfying hypercovder descent, a weak equivalence $\mathcal{F} \rightarrow \mathcal{G}$ is an objectwise equivalence.*

Proof. Let $f: \mathcal{F} \rightarrow \mathcal{G}$ be a weak equivalence of presheaves of symmetric spectra, both of which satisfy hypercovder descent. Without loss of generality, we can assume that \mathcal{F} and \mathcal{G} are both objectwise fibrant, and we look levelwise at the map of presheaves of simplicial sets $f_n: \mathcal{F}_n \rightarrow \mathcal{G}_n$. Then the canonical maps $\tilde{\pi}_q \mathcal{F}_n \rightarrow$

$\tilde{\pi}_{q-n}\mathcal{F}$ and $\tilde{\pi}_q\mathcal{G}_n \rightarrow \tilde{\pi}_{q-n}\mathcal{G}$ are isomorphisms for $q \geq 0$, and so $f_n: \mathcal{F}_n \rightarrow \mathcal{G}_n$ is a weak equivalence of presheaves of simplicial sets in the “universal model structure” of [18, 1.3.6.2]. Moreover, as products and homotopy limits of fibrant symmetric spectra are computed levelwise, hypercover descent for \mathcal{F} and \mathcal{G} imply hypercover descent for each \mathcal{F}_n and \mathcal{G}_n . Thus, \mathcal{F}_n and \mathcal{G}_n are fibrant objects in the universal model structure of [18, 1.3]. A weak equivalence of fibrant objects in this model structure is an objectwise weak equivalence. It follows that each $f_n: \mathcal{F}_n \rightarrow \mathcal{G}_n$ is an objectwise weak equivalence and hence that $f: \mathcal{F} \rightarrow \mathcal{G}$ is an objectwise levelwise weak equivalence. \square

Because homotopy limits commute, the following proposition is clear from the definition of hypercover descent.

Proposition 5.2.3. *Let F be a functor from a category \mathcal{D} to presheaves of symmetric spectra, and assume that for every object D in \mathcal{D} , $F(D)$ satisfies hypercover descent. Let \mathcal{F} be a presheaf of symmetric spectra and $f: \mathcal{F} \rightarrow F$ a natural transformation from the constant functor to the functor F . If f exhibits \mathcal{F} as the objectwise homotopy limit of F , then \mathcal{F} also satisfies hypercover descent.*

Every presheaf of symmetric spectra is functorially weakly equivalent to one that satisfies hypercover descent. Specifically, using the Jardine model structure on presheaves of symmetric spectra [31], the Jardine fibrant objects in particular are objectwise Ω -spectra where levelwise each presheaf of simplicial sets is globally fibrant in the sense of [29, §2] and therefore satisfies hypercover descent by [18, 1.2–1.3.6.2]. It follows that Jardine fibrant presheaves of symmetric spectra satisfy hypercover descent.

Proposition 5.2.4. *A Jardine fibrant presheaf of symmetric spectra satisfies hypercover descent. In particular, every presheaf of symmetric spectra admits a functorial weak equivalence to one that satisfies hypercover descent.*

The previous proposition and Proposition 5.2.2 show that the right derived functor of global sections can be calculated as the symmetric spectrum of global sections of a weakly equivalent presheaf that satisfies hypercover descent. We use the following terminology and notation.

Definition 5.2.5. For a presheaf of symmetric spectra \mathcal{F} on $A_{\text{ét}}$, let

$$\mathbb{H}(A_{\text{ét}}; \mathcal{F}) = R\mathcal{F}(A)$$

where $\mathcal{F} \rightarrow R\mathcal{F}$ is a weak equivalence and $R\mathcal{F}$ is a presheaf of symmetric spectra satisfying hypercover descent. We view $\mathbb{H}(A_{\text{ét}}; -)$ as a functor from the homotopy category of presheaves of symmetric spectra to the stable category and call $\mathbb{H}(A_{\text{ét}}; \mathcal{F})$ the *hypercohomology symmetric spectrum* associated to \mathcal{F} .

This definition disagrees in fundamental philosophy with the definition of “hypercohomology spectrum” introduced in Thomason [43, 1.33]; however, these definitions coincide under cohomological boundedness hypotheses on the sheaves of homotopy groups as we explain in the next subsection.

5.3. The Godement resolution. In this section, we review the definition of the Godement resolution of a presheaf of symmetric spectra, introduced by Thomason [43, 1.31]. For objectwise fibrant presheaves whose homotopy groups satisfy a cohomological boundedness hypothesis, the Godement construction provides a

Jardine fibrant replacement functor that is evidently a lax symmetric monoidal functor; in contrast, it is not currently known how to construct a lax symmetric monoidal fibrant replacement functor for any model structures on presheaves of symmetric spectra with the standard weak equivalences.

As in the Definition 5.1.2, for each point x in A , choose a corresponding geometric point $\bar{x}: \operatorname{spec} k(x)^s \rightarrow A$. Let \bar{A}_{disc} be the disjoint union of the $\operatorname{spec} k(x)^s$ and $\bar{A}_{\text{disc}} \rightarrow A$ the canonical map. Let $\mathcal{S}^{\bar{A}_{\text{disc}}}$ be the product of copies of the category of symmetric spectra indexed by the points of \bar{A}_{disc} (or equivalently, the points of A). Let p^* be the functor from presheaves of symmetric spectra to $\mathcal{S}^{\bar{A}_{\text{disc}}}$ that takes \mathcal{F} to the object given by its stalk in each coordinate, $\mathcal{F} \mapsto (\mathcal{F}_{\bar{x}})$. The functor p^* has a right adjoint p_* specified by the formula

$$p_*((T_{\bar{x}}))(V) := \prod_{y \in \bar{A}_{\text{disc}} \times_A V} T_{\pi(y)}.$$

where $(T_{\bar{x}})$ is an object in $\mathcal{S}^{\bar{A}_{\text{disc}}}$ and π denotes the projection $\bar{A}_{\text{disc}} \times_A V \rightarrow \bar{A}_{\text{disc}}$. The composite p_*p^* is then a comonad.

Definition 5.3.1. For \mathcal{F} a presheaf of symmetric spectra, define the *Godement resolution* of \mathcal{F} to be the augmented cosimplicial object $\mathcal{F} \rightarrow G^{\bullet}_{A_{\text{ét}}} \mathcal{F}$ (abbreviated to $G^{\bullet} \mathcal{F}$ when A is understood) given by the two-sided cobar construction

$$G^{\bullet} \mathcal{F} = \operatorname{CoB}^{\bullet}(p_*, p^*p_*, p^* \mathcal{F})$$

for the comonad p^*p_* acting on the left on the functor p^* and on the right on the functor p_* . We call $G_{A_{\text{ét}}} \mathcal{F} = \operatorname{Tot} G^{\bullet} \mathcal{F}$ (abbreviated to $G\mathcal{F}$ when A is understood) the *Godement construction* on \mathcal{F} .

The levels in the Godement resolution and the Godement construction are Jardine fibrant whenever \mathcal{F} is objectwise fibrant [29, 3.3]. The augmentation $\mathcal{F} \rightarrow G\mathcal{F}$ is not obviously a weak equivalence even when \mathcal{F} is objectwise fibrant, essentially because of the failure of the colimit defining stalks to commute with Tot in general; however, we do have the following proposition.

Proposition 5.3.2 ([29, 3.3], [43, 1.46]). *Let \mathcal{F} be an objectwise fibrant presheaf of symmetric spectra; then $G\mathcal{F}$ is Jardine fibrant and in particular satisfies hypercover descent. Assume in addition that there is a uniform bound on the cohomological dimension of $\tilde{\pi}_* \mathcal{F}$ for all objects in $A_{\text{ét}}$, i.e., there exists $N \geq 0$ such that $H^s_{\text{ét}}(V; \tilde{\pi}_t \mathcal{F}) = 0$ for all $s > N$, all t , and all V in $A_{\text{ét}}$; then the augmentation $\mathcal{F} \rightarrow G\mathcal{F}$ is a weak equivalence.*

Thomason [43, 1.33] defined the hypercohomology spectrum of an objectwise fibrant presheaf of spectra as the Godement construction. Under the hypotheses of the previous proposition, Thomason's definition agrees with Definition 5.2.5, but in general it may not. In the terminology of Thomason [43, 1.51], a presheaf of objectwise fibrant spectra \mathcal{F} satisfies “cohomological descent” if the augmentation map $\mathcal{F} \rightarrow G\mathcal{F}$ is an objectwise weak equivalence. It follows from the previous proposition that a presheaf of objectwise fibrant symmetric spectra that satisfies cohomological descent in this sense also satisfies hypercover descent.

Finally, it is clear from the formula that the left adjoint p^* is a strong symmetric monoidal functor and it follows that the right adjoint is a lax symmetric monoidal functor and that the unit $\operatorname{Id} \rightarrow p_*p^*$ and comultiplication $p_*p^* \rightarrow p_*p^*p_*p^*$ are lax

symmetric monoidal natural transformations. This implies the following proposition.

Proposition 5.3.3. *The Godement resolution and Godement construction are lax symmetric monoidal functors and the augmentations $\mathcal{F} \rightarrow G^\bullet \mathcal{F}$ and $\mathcal{F} \rightarrow G\mathcal{F}$ are lax symmetric monoidal natural transformations.*

5.4. Direct and inverse image functors. Given a map $f: A \rightarrow B$ of schemes, we obtain a direct image functor from the category of presheaves of symmetric spectra on $A_{\text{ét}}$ to the category of presheaves of symmetric spectra on $B_{\text{ét}}$

$$f_*: \mathcal{P}(A_{\text{ét}}, \mathcal{S}) \longrightarrow \mathcal{P}(B_{\text{ét}}, \mathcal{S})$$

and inverse image functor from the category of presheaves of symmetric spectra on $B_{\text{ét}}$ to the category of presheaves of symmetric spectra on $A_{\text{ét}}$

$$f^*: \mathcal{P}(B_{\text{ét}}, \mathcal{S}) \longrightarrow \mathcal{P}(A_{\text{ét}}, \mathcal{S})$$

defined (in the usual way) as follows. For a presheaf \mathcal{F} on A , the direct image $f_*\mathcal{F}$ is the presheaf on $B_{\text{ét}}$ defined by the formula

$$f_*\mathcal{F}(W) = \mathcal{F}(f^{-1}W)$$

for W in $B_{\text{ét}}$. For a presheaf \mathcal{G} on $B_{\text{ét}}$, the inverse image $f^*\mathcal{G}$ is the presheaf on $A_{\text{ét}}$ defined by the formula

$$\mathcal{G}(V) = \text{colim}_W \mathcal{G}(W)$$

for V in $A_{\text{ét}}$ where the colimit is over the category of maps $V \rightarrow W$ over B with W étale over B , i.e., the category of diagrams

$$\begin{array}{ccc} V & \longrightarrow & W \\ \downarrow & & \downarrow \\ A & \xrightarrow{f} & B \end{array}$$

with $V \rightarrow A$ the structure map and $W \rightarrow B$ the structure map for an object of $B_{\text{ét}}$. These functors form an adjoint pair with f^* the left and f_* the right adjoint.

For derived functors, exchange of colimits shows that f^* preserves all stalks

$$(f^*\mathcal{F})_{\bar{x}} \cong \mathcal{F}_{\overline{f(x)}},$$

and it follows that f^* preserves all weak equivalences. The left and right derived functor of f^* then exist and are formed just by applying the point-set functor. (We follow the topologists convention of using the same symbol to denote the derived functor in this context.) In general, f_* does not preserve weak equivalences, but it does preserve objectwise weak equivalences and so preserves weak equivalences between presheaves of symmetric spectra that satisfy hypercover descent. Thus, the right derived functor Rf_* is formed by applying f_* to a weakly equivalent object that satisfies hypercover descent. The adjunction is not a Quillen adjunction on the Jardine model structure; nevertheless, it induces an adjunction of derived functors.

Proposition 5.4.1. *Let $f: A \rightarrow B$ be a map of schemes. The inverse image functor $f^*: \mathcal{P}(B_{\text{ét}}, \mathcal{S}) \rightarrow \mathcal{P}(A_{\text{ét}}, \mathcal{S})$ preserves all weak equivalences of presheaves of symmetric spectra and the direct image functor $f_*: \mathcal{P}(A_{\text{ét}}, \mathcal{S}) \rightarrow \mathcal{P}(B_{\text{ét}}, \mathcal{S})$ preserves weak equivalences between presheaves of symmetric spectra that satisfy hypercover descent. The derived functors f^* and Rf_* are adjoint functors on the homotopy categories.*

Proof. The only thing remaining from the discussion above is to verify the adjunction on the homotopy category. The analogue of [18, 1.2] for presheaves of symmetric spectra would give a projective model structure with the standard weak equivalences, but where j^*, j_* would form a Quillen adjunction, cf. [18, 8.2]. Even without doing this work, using a Jardine fibrant replacement functor $\text{Id} \rightarrow J$, it is straightforward to construct unit and counit maps in the homotopy category making the derived functors of f^* and $f_* \circ J$ into adjoints. \square

As far as we know, Dugger-Hollander-Isaksen [18, 8.2] is the first paper to relate the hypercover descent property to the Quillen adjunction condition for f^*, f_* . The following additional result along these lines is clear from the construction of f_* .

Proposition 5.4.2. *Let $f: A \rightarrow B$ be a map of schemes. Then the direct image functor f_* takes presheaves of symmetric spectra that satisfy hypercover descent on $A_{\text{ét}}$ to presheaves of symmetric spectra that satisfy hypercover descent on $B_{\text{ét}}$.*

The functor f^* does not generally preserve hypercover descent.

Next we discuss the base change map. Given a commutative diagram of schemes

$$\begin{array}{ccc} A' & \xrightarrow{f'} & B' \\ a \downarrow & & \downarrow b \\ A & \xrightarrow{f} & B \end{array}$$

there is a canonical natural transformation

$$b^* f_* \mathcal{F} \rightarrow f'_* a^* \mathcal{F}$$

of presheaves of symmetric spectra on $B'_{\text{ét}}$ for presheaves of symmetric spectra \mathcal{F} on A defined as follows. For W' in $B'_{\text{ét}}$ the map takes the form

$$b^* f_* \mathcal{F}(W') = \text{colim}_W \mathcal{F}(f^{-1}(W)) \rightarrow \text{colim}_V \mathcal{F}(V) = f'_* a^* \mathcal{F}(W')$$

where the colimits are over the categories of diagrams

$$b^* f_* : \begin{array}{ccc} W' & \rightarrow & W \\ \downarrow & & \downarrow \\ B' & \xrightarrow{b} & B \end{array} \quad f'_* a^* : \begin{array}{ccc} (f')^{-1}(W') & \rightarrow & V \\ \downarrow & & \downarrow \\ A' & \xrightarrow{a} & A \end{array}$$

with $W \rightarrow B$ étale (on the left) and $V \rightarrow A$ étale (on the right). The map is induced by taking the diagram on the left to the corresponding diagram

$$\begin{array}{ccc} (f')^{-1}(W') & \rightarrow & f^{-1}(W) \\ \downarrow & & \downarrow \\ A' & \xrightarrow{a} & A \end{array}$$

on the right. Starting with a presheaf \mathcal{F} on A that satisfies hypercover descent and replacing $a^* \mathcal{F}$ by a presheaf that satisfies hypercover descent, we get a canonical natural transformation of composite derived functors.

Proposition 5.4.3. *Given a commutative diagram of schemes as above, there is a canonical natural transformation $b^* \circ Rf_* \rightarrow Rf'_* \circ a^*$.*

When the diagram is cartesian, the natural transformation above is called the *base change map*.

5.5. The K -theory presheaf. In this section, we discuss the construction of $K(1)$ -local algebraic K -theory with finite coefficients as a presheaf on the étale site of a scheme. There are various versions of such a construction in the literature, but for the work in later sections, a number of technical issues involving the multiplication, coherence, and fibrant approximation need to be addressed.

As usually constructed, algebraic K -theory is pseudofunctorial rather than functorial on schemes. There are several well-known ways of fixing this, the most popular being the “big vector bundle” approach of Grayson [23, §10] (see also [22, §C.4]). The category of big vector bundles on A has an exact symmetric monoidal tensor product. Applying Waldhausen’s iterated S_\bullet construction, we get a contravariant functor from schemes to the category of E_∞ ring symmetric spectra, as discussed in [4, §2]. We write this functor as K .

We can obtain a model for the functor $L_{KU}K$, localizing at the spectrum KU representing periodic complex K -theory, as follows. Let \mathbb{S}_{KU} be a cofibrant commutative ring symmetric spectrum model for the KU -local sphere spectrum. The functor $(-) \wedge \mathbb{S}_{KU}$ preserves weak equivalences and is a point-set model for the KU -localization functor, q.v. [8, 4.7]. Then $K(-) \wedge \mathbb{S}_{KU}$ is a contravariant functor from schemes to E_∞ ring symmetric spectra that represents KU -localized algebraic K -theory.

From the KU -localization, we can obtain the $K(1)$ -localization by p -completion, by which we will always mean localization with respect to the mod p Moore spectrum M_p . The p -completion of a spectrum T can be constructed as the homotopy inverse limit of the spectra T/p^n , where T/p^n is the cofiber of multiplication by p^n , or equivalently, is the derived smash of T with the mod p^n Moore spectrum M_{p^n} . For our results below, we use not just the $K(1)$ -localization, but the whole tower representing $\{L_{K(1)}K(-)/p^n\}$. Moreover, to simplify arguments, it is convenient to have it with a point-set multiplication. However, we do not need any unit or associativity properties of the multiplication. We therefore work in the context of (non-unital) “ A_2 ring spectra”.

For our purposes, we define an A_2 ring symmetric spectrum to be a symmetric spectrum T together with a map $T \wedge T \rightarrow T$, satisfying no further hypotheses. The category of A_2 ring symmetric spectra is evidently the category of algebras for a non- Σ operad A_2^{nu} in sets where $A_2^{nu}(0)$ is empty, $A_2^{nu}(1) = \{1\}$, and for $n \geq 2$, $A_2^{nu}(n)$ can be described in terms of binary trees with n leaves. The work of Elmendorf-Mandell [20, 1.3] provides a model structure on A_2 ring symmetric spectra with functorial factorization and in particular a fibrant replacement functor. Technically, the work there constructs a model structure induced from the positive stable model structure on symmetric spectra, but since A_2^{nu} comes as a non- Σ operad, the arguments there extend to construct a model structure induced from the usual stable model structure on symmetric spectra. It is useful to note that the cofibrant A_2 ring symmetric spectra are cofibrant as symmetric spectra, q.v. [20, 12.6]. We regard $K(-) \wedge \mathbb{S}_{KU}$ as a contravariant functor from schemes to A_2 ring symmetric spectra by neglect of structure.

Since p is odd, the mod p^n Moore spectrum admits a multiplication

$$M_{p^n} \wedge^L M_{p^n} \longrightarrow M_{p^n}$$

in the stable category, uniquely determined by inducing the usual multiplication $\mathbb{Z}/p^n \otimes \mathbb{Z}/p^n \rightarrow \mathbb{Z}/p^n$ on π_0 (see for example [43, A.6] for some references). The

multiplication is consistent with the system of maps

$$\cdots \longrightarrow M_{p^{n+1}} \longrightarrow M_{p^n} \longrightarrow \cdots \longrightarrow M_p$$

in the stable category and starting with $n = 1$, we can inductively choose cofibrant-fibrant A_2 ring symmetric spectra point-set models $M_{p^n}^{\text{Mod}}$ and a point-set tower of fibrations of A_2 ring spectra

$$\cdots \longrightarrow M_{p^{n+1}}^{\text{Mod}} \longrightarrow M_{p^n}^{\text{Mod}} \longrightarrow \cdots \longrightarrow M_p^{\text{Mod}}.$$

Smashing with $K(-) \wedge \mathbb{S}_{KU}$, we get a tower

$$\cdots \longrightarrow K(-) \wedge \mathbb{S}_{KU} \wedge M_{p^n}^{\text{Mod}} \longrightarrow \cdots \longrightarrow K(-) \wedge \mathbb{S}_{KU} \wedge M_p^{\text{Mod}}.$$

of contravariant functors from schemes to A_2 ring symmetric spectra representing the tower of ring spectra $\{L_{K(1)}K(-)/p^n\}$. Inductively, applying functorial factorization, we replace this with a weakly equivalent tower of functors to A_2 ring symmetric spectra where at each scheme the symmetric spectra are fibrant and the tower is a tower of fibrations.

Notation 5.5.1. Denote by \mathcal{K}/p^n and

$$\cdots \longrightarrow \mathcal{K}/p^{n+1} \longrightarrow \mathcal{K}/p^n \longrightarrow \cdots \longrightarrow \mathcal{K}/p$$

the contravariant functor from schemes to fibrant A_2 ring symmetric spectra and functorial tower of fibrations constructed above. Denote by \mathcal{K} the contravariant functor obtained as the inverse limit of the tower.

For a fixed scheme A , as constructed, $\mathcal{K}(A)$ is a fibrant point-set model for the ring spectrum $L_{K(1)}K(A)$ and $\mathcal{K}/p^n(A)$ is a fibrant point-set model for the ring spectrum $L_{K(1)}K(A)/p^n$. We then obtain presheaves of symmetric spectra $\mathcal{K}_{A_{\text{ét}}}$ and $\mathcal{K}_{A_{\text{ét}}}^{/p^n}$ on $A_{\text{ét}}$ by defining $\mathcal{K}_{A_{\text{ét}}}(V) = \mathcal{K}(V)$ and $\mathcal{K}_{A_{\text{ét}}}^{/p^n}(V) = \mathcal{K}/p^n(V)$. Thomason [43, 2.45 or 2.50] shows that under mild hypotheses on A , these presheaves satisfy “cohomological descent” (q.v. [43, 1.51]). It follows from Proposition 5.3.2 that under these hypotheses, $\mathcal{K}_{A_{\text{ét}}}$ and $\mathcal{K}_{A_{\text{ét}}}^{/p^n}$ also satisfy hypercover descent.

For a map of schemes $f: B \rightarrow A$, $L_{K(1)}K(A) \rightarrow L_{K(1)}K(B)$ is a well-defined map in the homotopy category of E_∞ ring spectra, and so its homotopy fiber $\text{Fib}(f)$ (in the stable category) canonically lifts to an object in the stable category of $L_{K(1)}K(A)$ -modules. From this structure, $\text{Fib}(f)/p^n$ obtains a canonical pairing in the stable category with $L_{K(1)}K(A)/p^n$. Using our point-set models $\mathcal{K}/p^n(A)$ and $\mathcal{K}/p^n(B)$, we obtain a natural point-set model for this pairing using the standard path object construction for the homotopy fiber.

Definition 5.5.2. Let $f: T \rightarrow T'$ be a map of symmetric spectra with T' levelwise fibrant. Let PT' denote the symmetric spectrum of based maps from the standard 1-simplex Δ^1 (based at 0) to T' , and let $ev: PT' \rightarrow T'$ be the map induced by evaluation at 1. The *standard path object construction* for the homotopy fiber of f , denoted $\text{Fib}(f)$, is the symmetric spectrum $T \times_{T'} PT'$ constructed as the fiber product of the maps f and ev .

Because of the hypothesis that T' is levelwise fibrant, the standard path object construction for the homotopy fiber of f gives a point-set model for the homotopy fiber of f .

Letting $\text{Fib}(f)^{/p^n} = \text{Fib}(\mathcal{K}^{/p^n}(A) \rightarrow \mathcal{K}^{/p^n}(B))$ be the standard path object construction for the homotopy fiber of the map $\mathcal{K}^{/p^n}(A) \rightarrow \mathcal{K}^{/p^n}(B)$, the symmetric spectrum $\text{Fib}(f)^{/p^n}$ now has a point set pairing

$$\mathcal{K}^{/p^n}(A) \wedge \text{Fib}(f)^{/p^n} \longrightarrow \text{Fib}(f)^{/p^n}$$

representing the pairing on $\text{Fib}(f)^{/p^n}$ in the stable category described above. Moreover, the $\text{Fib}(f)^{/p^n}$ fit into a tower of fibrations with limit $\text{Fib}(f)$ canonically isomorphic to $\text{Fib}(f) = \text{Fib}(\mathcal{K}(A) \rightarrow \mathcal{K}(B))$, which has a point-set pairing again representing the point-set pairing in the stable category described above. For the K -theoretic Tate-Poitou duality theorem, we apply this to the map

$$\kappa: \coprod \text{spec } F_\nu \longrightarrow \text{spec } \mathcal{O}_F[1/p].$$

6. $\text{Fib}(\kappa)$ AS HYPERCOHOMOLOGY AND $j_!$

As discussed above, the proof of the K -theoretic Tate-Poitou duality theorem relies on an étale hypercohomological interpretation of $\text{Fib}(\kappa)$. Arithmetic Tate-Poitou duality arises from a duality pairing plus a long exact sequence arising from *recollement*. The purpose of this section is to give a spectral lifting of this setup. We begin with a terse review.

For a fixed number field F and S the set of primes lying over p , let $Y = \text{spec } \mathcal{O}_F$, let U be the open subscheme $Y \setminus S = \text{spec } (\mathcal{O}_F[1/p])$, and let Z be the reduced closed subscheme $Y \setminus U = \coprod \text{spec } (\mathcal{O}_F/\nu)$. Writing i for the inclusion of Z in Y and j for the inclusion of U in Y , we have various adjoint functors on sheaves of abelian groups on the étale sites:

$$\begin{array}{ccc} \xleftarrow{i^*} & & \xleftarrow{j_!} \\ \mathcal{A}b(Z_{\text{ét}}) & \xrightarrow{i_*} \mathcal{A}b(Y_{\text{ét}}) & \xrightarrow{j^*} \mathcal{A}b(U_{\text{ét}}) \\ \xleftarrow{i^!} & & \xleftarrow{j_*} \end{array}$$

where each functor is the left adjoint of the functor below it. One consequence of *recollement* is that for any sheaf or complex of sheaves \mathcal{F} on $Y_{\text{ét}}$, the unit of the i_*, i^* adjunction and the counit of the $j_!, j^*$ adjunction fit into a short exact sequence

$$0 \longrightarrow j_! j^* \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow i_* i^* \mathcal{F} \longrightarrow 0.$$

Now we take \mathcal{F} to be a complex modeling $Rj_*(\mathbb{Z}/p^n(t))$, the total right derived functor of j_* applied to the locally constant sheaf $\mathbb{Z}/p^n(t)$ on $U_{\text{ét}}$. We can identify the terms in the resulting long exact sequence on hypercohomology as

$$(6.1) \quad \cdots \longrightarrow H^s(Y_{\text{ét}}; j_! \mathbb{Z}/p^n(t)) \longrightarrow H^s(U_{\text{ét}}; \mathbb{Z}/p^n(t)) \longrightarrow \prod_{\nu \in S} H_{\text{ét}}^s(F_\nu^h; \mathbb{Z}/p^n(t)) \longrightarrow \cdots$$

(cf. [34, II.2.3(a)]), where F_ν^h denotes the field of fractions of the henselization R_ν^h of the discrete valuation ring $(\mathcal{O}_F)_{(\nu)}$. F_ν^h consists of the elements in the completion F_ν that are algebraic over F . Because the inclusion of $F_\nu^h \rightarrow F_\nu$ induces an isomorphism of absolute Galois groups, it induces an isomorphism $H_{\text{ét}}^s(F_\nu^h; \mathbb{Z}/p^n(t)) \rightarrow H_{\text{ét}}^s(F_\nu; \mathbb{Z}/p^n(t))$.

Tate-Poitou duality is a consequence of the long exact sequence (6.1) and the perfect pairing [34, II.3.2–3]

$$(6.2) \quad H^s(U_{\text{ét}}; \mathbb{Z}/p^n(t)) \otimes H^{3-s}(Y_{\text{ét}}; j_! \mathbb{Z}/p^n(1-t)) \longrightarrow H^3(Y_{\text{ét}}; j_! \mathbb{Z}/p^n(1)) \\ \longrightarrow H^3(Y_{\text{ét}}, j_! \mathbb{G}_m) \cong \mathbb{Q}/\mathbb{Z}.$$

Here the isomorphism $H^3(Y_{\text{ét}}, j_! \mathbb{G}_m) \cong \mathbb{Q}/\mathbb{Z}$ is induced by the map from

$$C' = \text{coker}(H^2(U_{\text{ét}}; \mathbb{G}_m) \longrightarrow \prod H_{\text{ét}}^2(F_{\nu}^h; \mathbb{G}_m))$$

to $H^3(Y_{\text{ét}}, j_! \mathbb{G}_m)$ (in the corresponding long exact sequence for $\mathcal{F} = Rj_*(\mathbb{G}_m)$), which is an isomorphism, together with the canonical isomorphism from C' to

$$C = \text{coker}(H^2(U_{\text{ét}}; \mathbb{G}_m) \longrightarrow \prod H_{\text{ét}}^2(F_{\nu}; \mathbb{G}_m))$$

and the Albert-Brauer-Hasse-Noether isomorphism from C to \mathbb{Q}/\mathbb{Z} . In light of all this, the first step for K -theoretic Tate-Poitou duality is to identify $\text{Fib}(\kappa)$ in terms of a spectral version of $j_!$ applied to the $K(1)$ -local algebraic K -theory presheaf on $U_{\text{ét}}$.

For presheaves of symmetric spectra, we discussed in Subsection 5.4 the direct image and inverse image adjunctions j^* , j_* , and i^* , i_* . In this case, the direct image functor i_* commutes with stalks from Z and is trivial on stalks from the complement, and so i_* preserve all weak equivalences just as i^* and j^* do. We use the point-set functors as models for their derived functors. The functor j_* commutes with stalks from U , but does not always send a weak equivalence of presheaves of symmetric spectra to a weak equivalence on stalks in Z . For the derived functor of j_* , we must therefore restrict to presheaves that satisfy hypercover descent or replace presheaves by weakly equivalent presheaves that satisfies hypercover descent.

We can construct a model for the functor $j_!$ for presheaves of symmetric spectra as follows. First note that on sheaves of abelian groups $j_!$ is secretly a composite of two functors, the functor j_* and the functor $j_! j^*$; we will define $j_!$ for presheaves of symmetric spectra as an analogous composite functor, first defining a functor $j_!^*$ and then defining $j_!$ as the composite with the point-set functor j_* . Just as in sheaves of abelian groups where $j_!$ is exact while j_* is not, in presheaves of symmetric spectra, $j_!$ will preserve weak equivalences even where j_* does not. Assume that \mathcal{F} is objectwise levelwise fibrant; then so are the presheaves $i^* \mathcal{F}$ (on $Z_{\text{ét}}$) and $i_* i^* \mathcal{F}$ constructed above. We have a canonical map (the unit of the i^*, i_* adjunction) $\mathcal{F} \rightarrow i_* i^* \mathcal{F}$, and we define

$$j_!^* \mathcal{F} := \text{Fib}(\mathcal{F} \longrightarrow i_* i^* \mathcal{F}),$$

using the standard path object construction of the homotopy fiber (Definition 5.5.2) objectwise. The map $\mathcal{F} \rightarrow i_* i^* \mathcal{F}$ is a weak equivalence on the stalks \bar{x} in Z and the trivial map on stalks \bar{x} in U , and so $j_!^* \mathcal{F}$ has trivial stalks for \bar{x} in Z and the map on stalks $j_!^* \mathcal{F}_{\bar{x}} \rightarrow \mathcal{F}_{\bar{x}}$ is a weak equivalence for \bar{x} in U ; it follows that the sheaf of homotopy groups $\tilde{\pi}_* j_!^* \mathcal{F}$ is canonically isomorphic to $j_! j^* \tilde{\pi}_* \mathcal{F}$. Now for \mathcal{G} an objectwise levelwise fibrant presheaf of symmetric spectra on $U_{\text{ét}}$, the presheaf $j_* \mathcal{G}$ on $Y_{\text{ét}}$ obtained using the point-set functor j_* is objectwise levelwise fibrant, and we define

$$j_! \mathcal{G} := j_!^*(j_* \mathcal{G}).$$

We then again have that the homotopy sheaves $\tilde{\pi}_* j_! \mathcal{G}$ are canonically isomorphic to $j_! \tilde{\pi}_* \mathcal{G}$. We see (as previously mentioned) that $j_!$ preserves all weak equivalences between objectwise levelwise fibrant presheaves of symmetric spectra on $U_{\text{ét}}$, even those that the point-set functor j_* does not.

We now apply $j_!$ in the context of the $K(1)$ -local algebraic K -theory presheaves. We use the objectwise fibrant presheaf $\mathcal{K}_{U_{\text{ét}}}$ of symmetric spectra on $U_{\text{ét}}$ constructed in the previous section as the model for the $K(1)$ -local K -theory presheaf, and

likewise, when necessary, we use the objectwise fibrant presheaf $\mathcal{K}_{U_{\text{ét}}}^{/p^n}$ as the model for the $K(1)$ -local mod p^n K -theory. Since $\mathcal{K}_{U_{\text{ét}}}$ is in particular objectwise levelwise fibrant, we have the presheaf $j_!\mathcal{K}_{U_{\text{ét}}}$ on $Y_{\text{ét}}$ constructed above. Since $\mathcal{K}_{U_{\text{ét}}}$ satisfies hypercover descent (q.v. §5.5), we can use $j_*\mathcal{K}_{U_{\text{ét}}}$ as a model for the derived functor of j_* applied to $\mathcal{K}_{U_{\text{ét}}}$. We can now state the following theorem relating $j_!\mathcal{K}_{U_{\text{ét}}}$ with the homotopy fiber $\text{Fib}(\kappa)$ of the completion map. The statement is slightly complicated by the fact that $i_*i^*j_*\mathcal{K}_{U_{\text{ét}}}$ does not necessarily satisfy hypercover descent.

Theorem 6.3. *The p -completion of the hypercohomology symmetric spectrum $\mathbb{H}(Y_{\text{ét}}; j_!\mathcal{K}_{U_{\text{ét}}}(Y))$ is canonically isomorphic in the stable category to the homotopy fiber of the completion map*

$$L_{K(1)}K(\mathcal{O}_F[1/p]) \longrightarrow \prod_{\nu \in S} L_{K(1)}K(F_\nu).$$

The proof of Theorem 6.3 occupies the rest of the section. The strategy is to construct a canonical p -equivalence from $i^*j_*\mathcal{K}_{U_{\text{ét}}}$ to a p -complete objectwise fibrant presheaf \mathcal{F} of symmetric spectra on $Z_{\text{ét}}$ that satisfies hypercover descent. Then $i_*\mathcal{F}$ is an objectwise fibrant presheaf of symmetric spectra on $Y_{\text{ét}}$ that satisfies hypercover descent, which implies that the path object construction of the homotopy fiber $\mathcal{G} = \text{Fib}(j_*\mathcal{K}_{U_{\text{ét}}} \rightarrow i_*\mathcal{F})$ is also. The global section spectrum $\mathcal{G}(Y_{\text{ét}})$ then represents the p -completion of $\mathbb{H}(Y_{\text{ét}}; j_!\mathcal{K}_{U_{\text{ét}}})$ and by the construction of \mathcal{F} comes with a weak equivalence to our model $\text{Fib}(\kappa)$ from §5.5 for the homotopy fiber of the completion map on $K(1)$ -local algebraic K -theory.

To construct the presheaf \mathcal{F} in the outline above, we use the henselization R_ν^h of the discrete valuation ring $(\mathcal{O}_F)_{(\nu)}$ for ν a prime above p . Letting $Y' = \coprod \text{spec } R_\nu^h$, we can identify Z canonically as the set of closed points and $U' = Y' \setminus Z$ is canonically isomorphic to $\coprod \text{spec } F_\nu^h$, where (as above) F_ν^h denotes the field of fractions of R_ν^h . Write i' and j' for the inclusions and write $b: Y' \rightarrow Y$ and $a: U' \rightarrow U$ for the canonical maps. Let $\mathcal{F} = i'^*j'_*\mathcal{K}_{U'_{\text{ét}}}$ and more generally, let $\mathcal{F}/p^n = i'^*j'_*\mathcal{K}_{U'_{\text{ét}}}^{/p^n}$. We then have the composite map of presheaves of symmetric spectra on $Z_{\text{ét}}$

$$i^*j_*\mathcal{K}_{U_{\text{ét}}} \cong i'^*b^*j_*\mathcal{K}_{U_{\text{ét}}} \longrightarrow i'^*j'_*a^*\mathcal{K}_{U'_{\text{ét}}} \longrightarrow i'^*j'_*\mathcal{K}_{U'_{\text{ét}}} = \mathcal{F}$$

where the first map is induced by the identity $i = b \circ i'$, the second by the base change morphism, and the last map is the map $a^*\mathcal{K}_{U_{\text{ét}}} \rightarrow \mathcal{K}_{U'_{\text{ét}}}$ adjoint to the canonical map $\mathcal{K}_{U_{\text{ét}}} \rightarrow a_*\mathcal{K}_{U'_{\text{ét}}}$ obtained from the identification of both presheaves as restrictions of the functor \mathcal{K} . We likewise have a map of presheaves $i^*j_*\mathcal{K}_{U_{\text{ét}}}^{/p^n} \rightarrow \mathcal{F}/p^n$.

The functor i'^* has much better technical properties than the functor i^* , because for an object V in $Z_{\text{ét}}$ the category of diagrams

$$\begin{array}{ccc} V & \longrightarrow & W \\ \downarrow & & \downarrow \\ Z & \longrightarrow & Y' \end{array}$$

has an initial object $V_{Y'} \rightarrow Y'$ that can be constructed as follows. Since Z is the disjoint union of $\text{spec } \mathcal{O}_F/\nu$, V is the disjoint union of spec of fields, each of which is a finite separable extension of one of the finite fields \mathcal{O}_F/ν . We can take $V_{Y'}$ to be the disjoint union of spec of the extensions of R_ν^h that correspond to the extensions of \mathcal{O}_F/ν under the equivalence of the category of finite étale \mathcal{O}_F/ν -algebras with the category of finite étale R_ν^h -algebras [35, I.4.4]. Abstractly, we can see from this that

hypercover descent for $\mathcal{K}_{U'_{\text{ét}}}$ and $\mathcal{K}_{U'_{\text{ét}}}^{/p^n}$ which implies hypercover descent for $j'_*\mathcal{K}_{U'_{\text{ét}}}$ and $j'_*\mathcal{K}_{U'_{\text{ét}}}^{/p^n}$ implies hypercover descent for $\mathcal{F} = i'^*j'_*\mathcal{K}_{U'_{\text{ét}}}$ and $\mathcal{F}/p^n = i'^*j'_*\mathcal{K}_{U'_{\text{ét}}}^{/p^n}$. We can also see from this that the map from \mathcal{F} to the inverse limit of the \mathcal{F}/p^n is an isomorphism.

Concretely, the work of the last paragraph identifies

$$\mathcal{F}(V) \cong \mathcal{K}(V_{Y'} \times_{Y'} U'), \quad \mathcal{F}/p^n(V) \cong \mathcal{K}/p^n(V_{Y'} \times_{Y'} U').$$

We can now check that the map $i^*j_*\mathcal{K}_{U_{\text{ét}}}^{/p^n} \rightarrow \mathcal{F}/p^n$ is a weak equivalence. For V in $Z_{\text{ét}}$ connected, $V \cong \text{spec } k$ for a finite separable extension of \mathcal{O}_F/ν for some ν . We can then identify $V_{Y'} \times_{Y'} U'$ as the field of fractions $F_\nu^h(k)$ of the extension $R_\nu^h(k)$ of R_ν^h corresponding to k . It follows that $\mathcal{F}/p^n(V) \cong \mathcal{K}/p^n(\text{spec } F_\nu^h(k))$. On the other hand

$$i^*j_*\mathcal{K}_{U_{\text{ét}}}^{/p^n}(V) = \text{colim}_W \mathcal{K}/p^n(V \times_Y W).$$

with the colimit over maps $V \rightarrow W$ over Y with W étale over Y . The system of maps $\text{spec } F_\nu^h(k) \rightarrow V \times_Y W$ induce the map $i^*j_*\mathcal{K}_{U_{\text{ét}}}^{/p^n}(V) \rightarrow \mathcal{F}/p^n(V)$. In fact, $\text{spec } F_\nu^h(k)$ is the inverse limit of this system in the category of schemes (cf. [43, 1.42, 1.44]) and standard results [43, 1.43] on mod p^n K -theory then show that the map

$$\text{colim}_W \mathcal{K}/p^n(V \times_Y W) \longrightarrow \mathcal{K}/p^n(\text{spec } F_\nu^h(k))$$

is a weak equivalence. It follows that the map of presheaves $i^*j_*\mathcal{K}_{U_{\text{ét}}}^{/p^n}(V) \rightarrow \mathcal{F}/p^n(V)$ is a weak equivalence. Since derived smash product with the Moore spectra M_{p^n} commutes with homotopy colimits, each $i^*j_*\mathcal{K}_{U_{\text{ét}}}^{/p^n}$ is weakly equivalent to $i^*j_*\mathcal{K}_{U_{\text{ét}}}/p^n$, and we see that the map $i^*j_*\mathcal{K}_{U_{\text{ét}}} \rightarrow \mathcal{F}$ is a p -equivalence, i.e., induces a weak equivalence after derived smash with M_p .

Because i_* preserves weak equivalences and preserves the property of hypercover descent, $\text{Fib}(j_*\mathcal{K}_{U_{\text{ét}}}^{/p^n} \rightarrow i_*\mathcal{F}/p^n)$ satisfies hypercover descent and the induced map $j_!\mathcal{K}_{U_{\text{ét}}}^{/p^n} \rightarrow \text{Fib}(j_*\mathcal{K}_{U_{\text{ét}}}^{/p^n} \rightarrow i_*\mathcal{F}/p^n)$ is a weak equivalence. Likewise $\text{Fib}(j_*\mathcal{K}_{U_{\text{ét}}} \rightarrow \mathcal{F})$ satisfies hypercover descent and $j_!\mathcal{K}_{U_{\text{ét}}} \rightarrow \text{Fib}(j_*\mathcal{K}_{U_{\text{ét}}} \rightarrow i_*\mathcal{F})$ is a p -equivalence. The symmetric spectrum $\text{Fib}(j_*\mathcal{K}_{U_{\text{ét}}}^{/p^n} \rightarrow \mathcal{F}/p^n)(Y_{\text{ét}})$ therefore represents $\mathbb{H}(Y_{\text{ét}}; j_!\mathcal{K}_{U_{\text{ét}}}^{/p^n})$ and the symmetric spectrum $\text{Fib}(j_*\mathcal{K}_{U_{\text{ét}}} \rightarrow \mathcal{F})(Y_{\text{ét}})$ represents the p -completion of $\mathbb{H}(Y_{\text{ét}}; j_!\mathcal{K}_{U_{\text{ét}}})$.

By the work above we have a canonical isomorphism

$$i_*\mathcal{F}(Y_{\text{ét}}) \cong \mathcal{K}(\coprod \text{spec } F_\nu^h)$$

and so we obtain a canonical isomorphism

$$\text{Fib}(j_*\mathcal{K}_{U_{\text{ét}}} \rightarrow i_*\mathcal{F})(Y_{\text{ét}}) \cong \text{Fib}(\mathcal{K}(\text{spec } \mathcal{O}_F[1/p]) \rightarrow \mathcal{K}(\coprod \text{spec } F_\nu^h)).$$

The canonical map $\mathcal{K}(\coprod \text{spec } F_\nu^h)$ to $\mathcal{K}(\coprod \text{spec } F_\nu)$ is a weak equivalence by Thomason's theorem [43, 4.1] proving the $K(1)$ -local Quillen-Lichtenbaum conjecture. This completes the comparison to our standard model for the homotopy fiber of the completion map and finishes the proof of Theorem 6.3.

7. PROOF OF THE K -THEORETIC TATE-POITOU DUALITY THEOREM

In this section we prove the K -theoretic Tate-Poitou duality theorem. We deduce the result from the classical Tate-Poitou duality theorem; more precisely, we use the formulation in terms of Artin-Verdier duality (6.2). We argue in terms of a pairing of étale descent spectral sequences. Following Dugger [17, §4], we construct the descent spectral sequence using Whitehead towers, which have good multiplicative properties; this is a variant of the approach via Postnikov towers in Jardine [32, §6.1]. It is well-known that this spectral sequence is isomorphic to Thomason's descent spectral sequence from E_2 onward. We list the properties we need from the Whitehead tower in the following theorem.

Theorem 7.1. *There is a “Whitehead tower” functor W^\bullet from symmetric spectra to towers of symmetric spectra (indexed on \mathbb{Z}) and a consistent system of natural transformations $W^m \rightarrow \text{Id}$ with the following properties:*

- (i) *When T is objectwise fibrant, each $W^m T$ is a (non-fibrant) Ω -spectrum, i.e., the level fibrant replacement of $W^m T$ is fibrant in the stable model category of symmetric spectra.*
- (ii) *When T is objectwise fibrant, each map $W^m T \rightarrow T$ is an $(m-1)$ -connective cover, i.e., $\pi_q W^m T = 0$ for $q < m$ and $\pi_q W^m T \rightarrow \pi_q T$ is an isomorphism for $q \geq m$.*
- (iii) *W^\bullet comes with a pairing of the form*

$$W^{m_1} T_1 \wedge W^{m_2} T_2 \longrightarrow W^{m_1+m_2} (T_1 \wedge T_2)$$

which is natural in m_1, m_2, T_1, T_2 and commutes with the structure maps $W^\bullet T_1 \rightarrow T_1$, $W^\bullet T_2 \rightarrow T_2$, and $W^\bullet (T_1 \wedge T_2) \rightarrow T_1 \wedge T_2$.

We construct the Whitehead tower functor and verify its properties below in Section 8. The previous theorem provides the piece we need to construct the pairing on the descent spectral sequences.

Theorem 7.2. *The descent spectral sequences*

$$\begin{aligned} E_2^{s,t}(U_{\text{ét}}; \mathcal{K}_{U_{\text{ét}}}^{/p^n}) &= H^s(U_{\text{ét}}; \mathbb{Z}/p^n(t/2)) \implies \pi_{-s-t} \mathbb{H}(U_{\text{ét}}; \mathcal{K}_{U_{\text{ét}}}^{/p^n}) \quad \text{and} \\ E_2^{s,t}(Y_{\text{ét}}; j! \mathcal{K}_{U_{\text{ét}}}^{/p^n}) &= H^s(Y_{\text{ét}}; j! \mathbb{Z}/p^n(t/2)) \implies \pi_{-s-t} \mathbb{H}(Y_{\text{ét}}; j! \mathcal{K}_{U_{\text{ét}}}^{/p^n}) \end{aligned}$$

admit a pairing of the form

$$E_r^{s,t}(U_{\text{ét}}; \mathcal{K}_{U_{\text{ét}}}^{/p^n}) \otimes E_r^{s',t'}(Y_{\text{ét}}; j! \mathcal{K}_{U_{\text{ét}}}^{/p^n}) \longrightarrow E_r^{s+s',t+t'}(Y_{\text{ét}}; j! \mathcal{K}_{U_{\text{ét}}}^{/p^n})$$

which converges to the pairing

$$\pi_{-s-t} \mathbb{H}(U_{\text{ét}}; \mathcal{K}_{U_{\text{ét}}}^{/p^n}) \otimes \pi_{-s'-t'} \mathbb{H}(Y_{\text{ét}}; j! \mathcal{K}_{U_{\text{ét}}}^{/p^n}) \longrightarrow \pi_{-(s+s')-(t+t')} \mathbb{H}(Y_{\text{ét}}; j! \mathcal{K}_{U_{\text{ét}}}^{/p^n})$$

induced from the weak equivalences

$$\mathcal{K}^{/p^n}(\text{spec } \mathcal{O}_F[1/p]) \simeq \mathbb{H}(U_{\text{ét}}; \mathcal{K}_{U_{\text{ét}}}^{/p^n}), \quad \text{and} \quad \text{Fib}(\kappa)^{/p^n} \simeq \mathbb{H}(Y_{\text{ét}}; j! \mathcal{K}_{U_{\text{ét}}}^{/p^n})$$

and the pairing $\mathcal{K}^{/p^n}(\text{spec } \mathcal{O}_F[1/p]) \wedge^L \text{Fib}(\kappa)^{/p^n} \rightarrow \text{Fib}(\kappa)^{/p^n}$.

Proof. We start with the presheaf of symmetric spectra $\mathcal{K}_{U_{\text{ét}}}^{/p^n}$ on $U_{\text{ét}}$ and apply W^\bullet objectwise to obtain a tower of presheaves of symmetric spectra on $U_{\text{ét}}$. Applying a monoidal objectwise levelwise fibrant approximation functor L , then each

$LW^t(\mathcal{K}_{U_{\text{ét}}}^{/p^n})$ is objectwise fibrant. Applying the monoidal functor G , the Godement construction, each $GLW^t(\mathcal{K}_{U_{\text{ét}}}^{/p^n})$ satisfies hypercover descent, and writing

$$K^t(GLW^\bullet(\mathcal{K}_{U_{\text{ét}}}^{/p^n})) = \text{Fib}(GLW^t(\mathcal{K}_{U_{\text{ét}}}^{/p^n}) \longrightarrow GLW^{t-1}(\mathcal{K}_{U_{\text{ét}}}^{/p^n}))$$

for the path object construction of the homotopy fiber, $K^t(GLW^\bullet(\mathcal{K}_{U_{\text{ét}}}^{/p^n}))$ also satisfies hypercover descent. Since $\tilde{\pi}_t \mathcal{K}_{U_{\text{ét}}}^{/p^n} \cong \mathbb{Z}/p^n(t/2)$, we get a canonical isomorphism

$$\pi_{t-s} K^t(GLW^\bullet(\mathcal{K}_{U_{\text{ét}}}^{/p^n}))(U_{\text{ét}}) \cong H^s(U_{\text{ét}}; \mathbb{Z}/p^n(t/2)).$$

The tower $GLW^\bullet(\mathcal{K}_{U_{\text{ét}}}^{/p^n})(U_{\text{ét}})$ then in the standard way gives a spectral sequence (after the standard Whitehead/Postnikov Atiyah-Hirzebruch renumbering)

$$E_2^{s,t} = H^s(U_{\text{ét}}; \mathbb{Z}/p^n(t/2))$$

abutting to

$$\pi_{-s-t} \text{holim } GLW^\bullet(\mathcal{K}_{U_{\text{ét}}}^{/p^n})(U_{\text{ét}}).$$

Because $H^s(U_{\text{ét}}; \mathbb{Z}/p^n(t/2))$ is only non-zero in a finite range, the spectral sequence converges strongly, and moreover $\pi_{-s-t} \text{holim } GLW^\bullet(\mathcal{K}_{U_{\text{ét}}}^{/p^n})(U_{\text{ét}})$ is canonically isomorphic to $\pi_{-s-t} \mathcal{K}_{U_{\text{ét}}}^{/p^n}(U_{\text{ét}})$ by the comparison map (by Proposition 5.3.2). This is essentially a standard construction of the descent spectral sequence, cf. [32, §6.1], but because of the pairing property of W^\bullet above and monoidality of L and G , the spectral sequence is now multiplicative with multiplication on the E_2 -term induced by the cup product, converging to the multiplication on the homotopy groups of $\mathcal{K}_{U_{\text{ét}}}^{/p^n}(U_{\text{ét}})$ induced by the A_2 structure; see [16, 6.2] (plus Remark 8.6) for details of the construction of the pairing on the spectral sequence.

For the spectral sequence on $\text{Fib}(\kappa)$, we look at the map of towers (in \bullet) of presheaves of symmetric spectra on $Y_{\text{ét}}$

$$j_* GLW^\bullet \mathcal{K}_{U_{\text{ét}}}^{/p^n} \longrightarrow i_* i^* j_* GLW^\bullet \mathcal{K}_{U_{\text{ét}}}^{/p^n} \longrightarrow G(i_* i^* j_* GLW^\bullet \mathcal{K}_{U_{\text{ét}}}^{/p^n}).$$

The homotopy fiber of the first map is by definition $j_! GLW^\bullet \mathcal{K}_{U_{\text{ét}}}^{/p^n}$ and we denote the homotopy fiber of the composite map as \mathcal{X}^\bullet ; the comparison map $j_! GLW^\bullet \rightarrow \mathcal{X}^\bullet$ is a weak equivalence, with \mathcal{X}^\bullet satisfying hypercover descent. The fiber $K^t \mathcal{X}^\bullet$ of the map $\mathcal{X}^t \rightarrow \mathcal{X}^{t-1}$ then has homotopy groups

$$\pi_{-t-s} K^t \mathcal{X}(Y_{\text{ét}}) \cong H^s(Y_{\text{ét}}; j_! \tilde{\pi}_{-t} \mathcal{K}_{U_{\text{ét}}}^{/p^n})$$

(canonically) and the tower $\mathcal{X}^\bullet(Y_{\text{ét}})$ gives a spectral sequence

$$E_2^{s,t} = H^s(Y_{\text{ét}}; j_! \mathbb{Z}/p^n(t/2))$$

abutting to

$$\pi_{-s-t} \text{holim } \mathcal{X}^\bullet(Y_{\text{ét}}).$$

Again because $H^s(Y_{\text{ét}}; j_! \mathbb{Z}/p^n(t/2))$ is only non-zero in a finite range, the spectral sequence converges strongly, and $\pi_{-s-t} \text{holim } \mathcal{X}^\bullet(Y_{\text{ét}})$ is canonically isomorphic to $\pi_{-s-t} \text{Fib}(\kappa)$ by the comparison map. The pairing property of W^\bullet induces a pairing

$$G_{U_{\text{ét}}} LW^t \mathcal{K}_{U_{\text{ét}}}^{/p^n}(U_{\text{ét}}) \wedge \mathcal{X}^{t'}(Y_{\text{ét}}) = j_* G_{Y_{\text{ét}}} LW^t \mathcal{K}_{U_{\text{ét}}}^{/p^n}(Y_{\text{ét}}) \wedge \mathcal{X}^{t'}(Y_{\text{ét}}) \longrightarrow \mathcal{X}^{t+t'}(Y_{\text{ét}}),$$

inducing a pairing of spectral sequences.

Finally we need to compare this pairing with our standard model

$$\mathcal{K}^{/p^n}(\text{spec } \mathcal{O}_F[1/p]) \wedge \text{Fib}(\kappa)^{/p^n} \longrightarrow \text{Fib}(\kappa)^{/p^n}$$

for the pairing of the $K(1)$ -local mod p^n algebraic K -theory and the fiber. But this is straightforward using the weak equivalence $i^* j_* \mathcal{K}_{U_{\text{ét}}}^{/p^n} \rightarrow i'^* j'_* \mathcal{K}_{U'_{\text{ét}}}^{/p^n} (= \mathcal{F}/p^n)$ of the previous section (in the notation there). \square

Next we need to identify the multiplication on the E_2 -term, which takes the form

$$H^s(U_{\text{ét}}; \mathbb{Z}/p^n(t/2)) \otimes H^{s'}(Y_{\text{ét}}; j_! \mathbb{Z}/p^n(t'/2)) \longrightarrow H^{s+s'}(Y_{\text{ét}}; j_! \mathbb{Z}/p^n(t/2 + t'/2)).$$

In the notation of the proof of Theorem 7.2, the multiplication is induced by the map

$$j_* K^t(GLW^\bullet(\mathcal{K}_{U_{\text{ét}}}^{/p^n})(Y_{\text{ét}})) \wedge K^{t'} \mathcal{X}^\bullet(Y_{\text{ét}}) \longrightarrow K^{t+t'} \mathcal{X}^\bullet(Y_{\text{ét}}).$$

By construction $K^t \mathcal{X}^\bullet(Y_{\text{ét}})$ is a model for the hypercohomology spectrum

$$\mathbb{H}(Y_{\text{ét}}; \Sigma^t j_! H\mathbb{Z}/p^n(t/2))$$

and $j_* K^{t'}(GLW^\bullet(\mathcal{K}_{U_{\text{ét}}}^{/p^n})(Y_{\text{ét}}))$ is a model for the hypercohomology spectrum

$$\mathbb{H}(Y_{\text{ét}}; \Sigma^{t'} Rj_* H\mathbb{Z}/p^n(t'/2)),$$

where we have written $H\mathcal{F}$ for an objectwise fibrant model of the Eilenberg-Mac Lane presheaf of symmetric spectra on the sheaf of abelian groups \mathcal{F} . Thus, we can identify the induced map on homotopy groups as the composite of the cup product

$$\begin{aligned} H^*(U_{\text{ét}}; \mathbb{Z}/p^n(t'/2)) \otimes H^*(Y_{\text{ét}}; j_! \mathbb{Z}/p^n(t/2)) \\ \cong \pi_{-*} \mathbb{H}(Y_{\text{ét}}; Rj_* H\mathbb{Z}/p^n(t'/2)) \otimes \pi_{-*} \mathbb{H}(Y_{\text{ét}}; j_! H\mathbb{Z}/p^n(t/2)) \\ \longrightarrow \pi_{-*}(\mathbb{H}(Y_{\text{ét}}; Rj_* H\mathbb{Z}/p^n(t'/2)) \wedge^L \mathbb{H}(Y_{\text{ét}}; j_! H\mathbb{Z}/p^n(t/2))) \\ \longrightarrow \pi_{-*}(\mathbb{H}(Y_{\text{ét}}; j_! H\mathbb{Z}/p^n(t/2) \wedge^L Rj_* H\mathbb{Z}/p^n(t'/2))) \end{aligned}$$

and the map on hypercohomology spectra induced by a map

$$Rj_* H\mathbb{Z}/p^n(t'/2) \wedge^L j_! H\mathbb{Z}/p^n(t/2) \longrightarrow j_! H\mathbb{Z}/p^n(t/2 + t'/2) \simeq H j_! \mathbb{Z}/p^n(t/2 + t'/2)$$

induced by the pairing. We have a canonical isomorphism in the stable category of presheaves of symmetric spectra

$$Rj_* H\mathbb{Z}/p^n(t/2) \wedge^L j_! H\mathbb{Z}/p^n(t'/2) \simeq j_!(H\mathbb{Z}/p^n(t/2) \wedge^L H\mathbb{Z}/p^n(t'/2))$$

and since the target

$$j_! H\mathbb{Z}/p^n(t/2 + t'/2) \simeq H j_! \mathbb{Z}/p^n(t/2 + t'/2)$$

is an Eilenberg-Mac Lane presheaf, the map is determined by the restriction to the connective cover

$$j_!(H\mathbb{Z}/p^n(t/2) \wedge^L H\mathbb{Z}/p^n(t'/2))[0, \infty) \simeq j_!(H\mathbb{Z}/p^n(t/2 + t'/2)).$$

By looking at stalks, we see that the self-map of $j_!(H\mathbb{Z}/p^n(t/2 + t'/2))$ is the identity. As a consequence, it follows that the map

$$H^*(U_{\text{ét}}; \mathbb{Z}/p^n(t/2)) \otimes H^*(Y_{\text{ét}}; j_! \mathbb{Z}/p^n(t'/2)) \longrightarrow H^*(Y_{\text{ét}}; j_! \mathbb{Z}/p^n(t/2 + t'/2))$$

on the E_2 -term in Theorem 7.2 factors through the corresponding cup product map in the derived category of sheaves of abelian groups on $Y_{\text{ét}}$,

$$\begin{aligned}
H^*(U_{\text{ét}}; \mathbb{Z}/p^n(t/2)) \otimes H^*(Y_{\text{ét}}; j_! \mathbb{Z}/p^n(t'/2)) \\
&= \mathbb{H}^*(Y_{\text{ét}}; Rj_* \mathbb{Z}/p^n(t/2)) \otimes H^*(Y_{\text{ét}}; j_! \mathbb{Z}/p^n(t'/2)) \\
&\longrightarrow H^*(\mathbb{H}_{\mathcal{A}b}(Y_{\text{ét}}; Rj_* \mathbb{Z}/p^n(t/2)) \otimes^L \mathbb{H}_{\mathcal{A}b}(Y_{\text{ét}}; j_! \mathbb{Z}/p^n(t'/2))) \\
&\longrightarrow \mathbb{H}^*(Y_{\text{ét}}; Rj_* \mathbb{Z}/p^n(t/2) \otimes^L j_! \mathbb{Z}/p^n(t'/2)) \\
&\cong \mathbb{H}^*(Y_{\text{ét}}; j_! (\mathbb{Z}/p^n(t/2) \otimes^L \mathbb{Z}/p^n(t'/2))) \\
&\longrightarrow H^*(Y_{\text{ét}}; j_! (\mathbb{Z}/p^n(t/2) \otimes \mathbb{Z}/p^n(t'/2))) \\
&\cong H^*(Y_{\text{ét}}; j_! \mathbb{Z}/p^n(t/2 + t'/2)).
\end{aligned}$$

Here we have written $\mathbb{H}_{\mathcal{A}b}(Y_{\text{ét}}; -)$ for the hypercohomology object of a sheaf of abelian groups (an object in the derived category of abelian groups) and $\mathbb{H}^*(Y_{\text{ét}}; -)$ for its hypercohomology groups $H^*(\mathbb{H}_{\mathcal{A}b}(Y_{\text{ét}}; -))$. While this identifies the multiplication, it is not precisely the identification we need; the identification we need is given in the following proposition.

Proposition 7.3. *Under the canonical isomorphism*

$$\begin{aligned}
H^*(U_{\text{ét}}; \mathbb{Z}/p^n(t/2)) &\cong \text{Ext}_{U_{\text{ét}}}^*(\mathbb{Z}/p^n(t'/2), \mathbb{Z}/p^n(t/2 + t'/2)) \\
&\cong \text{Ext}_{Y_{\text{ét}}}^*(j_! \mathbb{Z}/p^n(t'/2), j_! \mathbb{Z}/p^n(t/2 + t'/2)),
\end{aligned}$$

the multiplication on the E_2 -term in Theorem 7.2

$$H^*(U_{\text{ét}}; \mathbb{Z}/p^n(t/2)) \otimes H^*(Y_{\text{ét}}; j_! \mathbb{Z}/p^n(t'/2)) \longrightarrow H^*(Y_{\text{ét}}; j_! \mathbb{Z}/p^n(t/2 + t'/2))$$

coincides with the Yoneda pairing

$$\begin{aligned}
\text{Ext}_{Y_{\text{ét}}}^*(j_! \mathbb{Z}/p^n(t'/2), j_! \mathbb{Z}/p^n(t/2 + t'/2)) \otimes H^*(Y_{\text{ét}}; j_! \mathbb{Z}/p^n(t'/2)) \\
\longrightarrow H^*(Y_{\text{ét}}; j_! \mathbb{Z}/p^n(t/2 + t'/2))
\end{aligned}$$

Proof. Given the identification above of the multiplication on the E_2 term in terms of the cup product, the proposition now follows from the basic relationship between the cup product and the Yoneda product in the derived category of sheaves of abelian groups on $Y_{\text{ét}}$, cf. [35, §5.1]: For sheaves of abelian groups \mathcal{F} and \mathcal{G} , the following diagram in the derived category commutes

$$\begin{array}{ccc}
\mathbb{H}_{\mathcal{A}b}(Y_{\text{ét}}; R\mathcal{H}om(\mathcal{F}, \mathcal{G})) \otimes^L \mathbb{H}_{\mathcal{A}b}(Y_{\text{ét}}; \mathcal{F}) & \longrightarrow & \mathbb{H}_{\mathcal{A}b}(Y_{\text{ét}}; R\mathcal{H}om(\mathcal{F}, \mathcal{G}) \otimes^L \mathcal{F}) \\
\cong \downarrow & & \downarrow \\
R\mathcal{H}om(\mathcal{F}, \mathcal{G}) \otimes^L \mathbb{H}_{\mathcal{A}b}(Y_{\text{ét}}; \mathcal{F}) & \longrightarrow & \mathbb{H}_{\mathcal{A}b}(Y_{\text{ét}}; \mathcal{G})
\end{array}$$

where the top arrow is the cup product and the bottom arrow and righthand arrows are the appropriate evaluation maps. \square

We now have the ingredients we need to prove the K -theoretic Tate-Poitou duality theorem.

Proof of the K -theoretic Tate-Poitou duality theorem. The pairing

$$\mathcal{K}(\text{spec } \mathcal{O}_F[1/p]) \wedge^L \text{Fib}(\kappa) \longrightarrow \text{Fib}(\kappa)$$

and the canonical map

$$u_{\mathcal{O}_F} : \text{Fib}(\kappa) \longrightarrow \Sigma^{-1} I_{\mathbb{Z}_p} \mathbb{S}$$

of (1.5) give a pairing

$$\mathcal{K}(\text{spec } \mathcal{O}_F[1/p]) \wedge^L \text{Fib}(\kappa) \longrightarrow \Sigma^{-1} I_{\mathbb{Z}_p} \mathbb{S},$$

which induces a map

$$(7.4) \quad \text{Fib}(\kappa) \longrightarrow \Sigma^{-1} I_{\mathbb{Z}_p} (\mathcal{K}(\text{spec } \mathcal{O}_F[1/p])) \simeq \Sigma^{-1} I_{\mathbb{Q}/\mathbb{Z}} (\mathcal{K}(\text{spec } \mathcal{O}_F[1/p]) \wedge^L M_{\mathbb{Q}_p/\mathbb{Z}_p}).$$

We need to see that it is a weak equivalence. Since both sides are p -complete, it suffices to check that (7.4) becomes a weak equivalence after smashing with the mod p Moore spectrum M_p on both sides. Then we are looking at the map

$$(7.5) \quad \text{Fib}(\kappa)/p \longrightarrow \Sigma^{-1} I_{\mathbb{Q}/\mathbb{Z}} (\mathcal{K}(\text{spec } \mathcal{O}_F[1/p]) \wedge^L M_{\mathbb{Q}_p/\mathbb{Z}_p})/p \simeq \Sigma^{-1} I_{\mathbb{Q}/\mathbb{Z}} (\mathcal{K}(\text{spec } \mathcal{O}_F[1/p])/p),$$

which is adjoint to a map

$$(7.6) \quad \mathcal{K}(\text{spec } \mathcal{O}_F[1/p])/p \wedge^L \text{Fib}(\kappa)/p \longrightarrow \Sigma^{-1} I_{\mathbb{Q}/\mathbb{Z}} \mathbb{S}.$$

Naturality and the fact that for an odd prime the map

$$M_p \wedge^L M_p \simeq F(M_p, M_{\mathbb{Q}_p/\mathbb{Z}_p}) \wedge^L M_p \longrightarrow M_{\mathbb{Q}_p/\mathbb{Z}_p}$$

induced by evaluation is the same as the composite of the multiplication on M_p and the inclusion of M_p in $M_{\mathbb{Q}_p/\mathbb{Z}_p}$ imply that the map (7.6) is the composite of the multiplication

$$\mathcal{K}(\text{spec } \mathcal{O}_F[1/p])/p \wedge^L \text{Fib}(\kappa)/p \longrightarrow \text{Fib}(\kappa)/p$$

and the map

$$\text{Fib}(\kappa)/p \longrightarrow \text{Fib}(\kappa) \wedge^L M_{\mathbb{Q}_p/\mathbb{Z}_p} \longrightarrow \Sigma^{-1} I_{\mathbb{Q}/\mathbb{Z}} \mathbb{S}$$

(where $\text{Fib}(\kappa) \wedge^L M_{\mathbb{Q}_p/\mathbb{Z}_p} \rightarrow \Sigma^{-1} I_{\mathbb{Q}/\mathbb{Z}} \mathbb{S}$ is the map 1.4 adjoint to $u_{\mathcal{O}_F}$). Because (6.2) is a perfect pairing, Theorem 7.2 and Proposition 7.3 imply that (7.6) induces a perfect pairing on homotopy groups

$$\pi_q(\mathcal{K}(\text{spec } \mathcal{O}_F[1/p])/p) \otimes \pi_{-1-q}(\text{Fib}(\kappa)/p) \longrightarrow \mathbb{Q}/\mathbb{Z}.$$

This implies that (7.5) is a weak equivalence, and we conclude that (7.4) is a weak equivalence. \square

In the proof above, we implicitly used the following straightforward result to deduce a perfect pairing on homotopy groups from the perfect pairing on the E_∞ -term of a spectral sequence.

Proposition 7.7. *Let*

$$\begin{aligned} 0 &= F^{-1}A \subset F^0A \subset F^1A \subset F^2A \subset F^3A = A, \\ 0 &= F^{-1}B \subset F^0B \subset F^1B \subset F^2B \subset F^3B = B \end{aligned}$$

be filtered abelian groups, and let $F^\mathbb{Q}/\mathbb{Z}$ be the filtered abelian group that is \mathbb{Q}/\mathbb{Z} in filtration level 3 and above and 0 in filtration levels 2 and below. Let $F^sA \otimes F^tB \rightarrow F^{s+t}\mathbb{Q}/\mathbb{Z}$ be a filtered homomorphism. If the associated graded pairing $Gr^*A \otimes Gr^{3-*}B \rightarrow \mathbb{Q}/\mathbb{Z}$ is perfect then so is the pairing $A \otimes B \rightarrow \mathbb{Q}/\mathbb{Z}$.*

Proof. As a base case, we have that the pairing of filtered abelian groups induces a perfect pairing of $Gr^0 A = F^0 A$ and $Gr^3 B = B/F^2 B$. By induction, assume that the pairing of filtered abelian groups induces a perfect pairing of $F^n A$ and $B/F^{2-n} B$, and consider the map

$$F^{n+1} A \otimes B \longrightarrow \mathbb{Q}/\mathbb{Z}.$$

Since $F^2 \mathbb{Q}/\mathbb{Z}$ is zero, the restriction to $F^{n+1} A \otimes F^{2-(n+1)} B$ is the zero map, and so the previous map factors as

$$F^{n+1} A \otimes B/F^{2-(n+1)} B \longrightarrow \mathbb{Q}/\mathbb{Z}.$$

The adjoint map $F^{n+1} A \rightarrow \text{Hom}(B/F^{2-(n+1)} B, \mathbb{Q}/\mathbb{Z})$ restricts (on subgroups) to an isomorphism

$$F^n A \longrightarrow \text{Hom}(B/F^{2-n} B, \mathbb{Q}/\mathbb{Z}) = \text{Hom}(B/F^{3-(n+1)} B, \mathbb{Q}/\mathbb{Z})$$

by the inductive hypothesis, and induces (on quotients) an isomorphism

$$\begin{aligned} F^{n+1} A/F^n A &\longrightarrow \text{Hom}(B/F^{2-(n+1)} B, \mathbb{Q}/\mathbb{Z}) / \text{Hom}(B/F^{3-(n+1)} B, \mathbb{Q}/\mathbb{Z}) \\ &= \text{Hom}(F^{3-(n+1)} B/F^{2-(n+1)} B, \mathbb{Q}/\mathbb{Z}) \end{aligned}$$

by the hypothesis of the theorem, and so is itself an isomorphism by the Five Lemma. \square

Remark 7.8. Although not in the statement of Theorem 7.2, the proof gives an argument that the descent spectral sequence is multiplicative, with multiplication on the E_2 -term induced by the cup product in étale cohomology and converging (under the usual hypotheses) to the multiplication on $K(1)$ -local algebraic K -theory. This can also be proved directly using Thomason's descent spectral sequence (using the Godement resolution without the Whitehead tower) and the analysis of the homotopy group spectral sequence of cosimplicial resolution of Bousfield-Kan [9]. The comparison of Atiyah-Hirzebruch spectral sequences in [24, Appendix B] adapts to prove that the Whitehead tower construction of the descent spectral sequence and the Godement resolution construction of the spectral sequence are isomorphic from E_2 onward.

8. A MULTIPLICATIVE WHITEHEAD TOWER FUNCTOR FOR SYMMETRIC SPECTRA

In this section, we establish Theorem 7.1 by constructing the multiplicative Whitehead tower functor. The basic construction is as follows.

Construction 8.1. For each triple of non-negative integers i, j, k , let $\phi_{i,j,k}$ be the map of symmetric spectra

$$\phi_{i,j,k}: F_i \text{sd}^j \partial \Delta_+^k \longrightarrow F_i \text{sd}^j \Delta_+^k$$

where F_i is the left adjoint to evaluation at the i th spot, $\partial \Delta^k \rightarrow \Delta^k$ is the inclusion of the boundary in the standard k -simplex, and sd^j is the j th iterate of the barycentric subdivision functor on simplicial complexes (with sd^0 understood to be the identity). For shorthand, we also write $A_{i,j,k}$ and $B_{i,j,k}$ for domain and codomain (respectively), and as base case for an induction, we let $\phi_{(1;((i,j,k)))} = \phi_{i,j,k}$, $A_{(1;((i,j,k)))} = A_{i,j,k}$, and $B_{(1;((i,j,k)))} = B_{i,j,k}$. For each positive integer n and

each n -tuple of triples of non-negative integers $\vec{s} = ((i_1, j_1, k_1), \dots, (i_n, j_n, k_n))$, inductively define $\phi_{(n; \vec{s})}$ to be the map

$$\begin{aligned} \phi_{(n-1; \vec{r})} \square \phi_{i_n, j_n, k_n} : A_{(n-1; \vec{r})} \wedge B_{i_n, j_n, k_n} \cup_{A_{(n-1; \vec{r})} \wedge A_{i_n, j_n, k_n}} B_{(n-1; \vec{r})} \wedge A_{i_n, j_n, k_n} \\ \longrightarrow B_{(n-1; \vec{r})} \wedge B_{i_n, j_n, k_n} \end{aligned}$$

where $\vec{r} = ((i_1, j_1, k_1), \dots, (i_{n-1}, j_{n-1}, k_{n-1}))$. We let I_S be the set indexing these ordered pairs $(n; \vec{s})$ (for all n). Then each element $(n; \vec{s})$ of I_S indexes a “cell” $\phi_{(n; \vec{s})}$ of formal dimension $\sum(k_\ell - i_\ell) \in \mathbb{Z}$. Let I_S^d denote the subset of I_S indexing the cells of formal dimension d .

Given a symmetric spectrum T , define the symmetric spectrum $W^m T$ and map $W^m T \rightarrow T$ as follows: Let $W_\ell^m T = *$ for $\ell < m$. For $\ell \geq m$, inductively we have $W_{\ell-1}^m T \rightarrow T$, and we let $W_\ell^m T \rightarrow T$ be the symmetric spectrum and map obtained from the pushout in the commutative diagram

$$\begin{array}{ccc} \bigvee_{(n; \vec{s}) \in I_S^\ell} \bigvee_{\alpha} A_{(n; \vec{s})} & \xrightarrow{\bigvee \bigvee \phi_{(n; \vec{s})}} & \bigvee_{(n; \vec{s}) \in I_S^\ell} \bigvee_{\alpha} B_{(n; \vec{s})} \\ \bigvee \bigvee g_\alpha \downarrow & & \downarrow \bigvee \bigvee h_\alpha \\ W_{\ell-1}^m T & \xrightarrow{\quad} & T \end{array}$$

where the outer wedge is indexed over the set I_S^ℓ indexing the cells of formal dimension ℓ and (for each fixed $(n; \vec{s}) \in I_S^\ell$) the inner wedge α is indexed on the set of commutative diagrams

$$\begin{array}{ccc} A_{(n; \vec{s})} & \xrightarrow{\phi_{(n; \vec{s})}} & B_{(n; \vec{s})} \\ g \downarrow & & \downarrow h \\ W_{\ell-1}^m T & \xrightarrow{\quad} & T \end{array}$$

Let $W^m T = \text{colim } W_\ell^m T$ and let $W^m T \rightarrow T$ be the induced map.

The following is clear from the construction and the simplicial approximation theorem.

Proposition 8.2. *W^m is an endofunctor on symmetric spectra and $W^m \rightarrow \text{Id}$ is a natural transformation. For each symmetric spectrum T and each $i \geq 0$, the map on i th spaces $W^m T(i) \rightarrow T(i)$ is an $(m+i-1)$ -connected cover. In other words, if $m+i < 1$, then $W^m T(i) \rightarrow T(i)$ is a weak equivalence; if $m+i \geq 1$, then $W^m T(i)$ is $(m+i-1)$ -connected and the map $W^m T(i) \rightarrow T(i)$ induces an isomorphism on π_q for $q \geq m+i$.*

Next we extend Construction 8.1 to a coaugmented tower.

Construction 8.3. For $m \in \mathbb{Z}$, let $W^{m+1} \rightarrow W^m$ be the natural transformation constructed inductively as follows. For $\ell \leq m$, $W_\ell^{m+1} T = *$ and the map $W_\ell^{m+1} T \rightarrow W_\ell^m T$ is the unique map. Having constructed $W_\ell^{m+1} T \rightarrow W_\ell^m T$ consistent with

the augmentation to T , then every cell used to build $W_{\ell+1}^{m+1}T$ from $W_{\ell}^{m+1}T$,

$$\begin{array}{ccccc} A_{(n;\vec{s})} & \xrightarrow{\phi_{(n;\vec{s})}} & B_{(n;\vec{s})} & & \\ \downarrow & & \downarrow & & \\ W_{\ell}^{m+1}T & \longrightarrow & W_{\ell}^m T & \longrightarrow & T \end{array}$$

specifies a corresponding cell used to build $W_{\ell+1}^m T$ from $W_{\ell}^m T$. Let $W_{\ell+1}^{m+1}T \rightarrow W_{\ell+1}^m T$ be the induced map on pushouts and $W_{\ell+1}^{m+1}T \rightarrow W_{\ell+1}^m T$ the induced map on colimits.

Next we construct the pairing.

Construction 8.4. Given symmetric spectra T_1, T_2 and $m_1, m_2 \in \mathbb{Z}$, construct the natural map

$$W^{m_1}T_1 \wedge W^{m_2}T_2 \longrightarrow W^{m_1+m_2}(T_1 \wedge T_2)$$

as follows. For $\ell \in \mathbb{Z}$, let

$$F^{\ell} = \operatorname{colim}_{\ell_1+\ell_2 \leq \ell} W_{\ell_1}^{m_1}T_1 \wedge W_{\ell_2}^{m_2}T_2.$$

For ℓ_1, ℓ_2 with $\ell_1 < m_1$ or $\ell_2 < m_2$, $W_{\ell_1}^{m_1}T_1 \wedge W_{\ell_2}^{m_2}T_2 = *$, so in particular $F^{\ell} = *$ for $\ell < m_1 + m_2$. For $\ell \geq m_1 + m_2$, F^{ℓ} is the pushout

$$\begin{array}{ccc} \bigvee_{\ell_1+\ell_2=\ell} \bigvee_{\substack{(n_1, \vec{s}_1) \in I_S^{\ell_1} \\ (n_2, \vec{s}_2) \in I_S^{\ell_2}}} \bigvee_{\alpha_1, \alpha_2} \operatorname{dom}(\phi_{(n_1, \vec{s}_1)} \square \phi_{(n_2, \vec{s}_2)}) & \longrightarrow & \bigvee \bigvee \bigvee \operatorname{cod}(\phi_{(n_1, \vec{s}_1)} \square \phi_{(n_2, \vec{s}_2)}) \\ \downarrow & & \downarrow \\ F^{\ell-1} & \longrightarrow & F^{\ell} \end{array}$$

where (for fixed $(n_1, \vec{s}_1) \in I_S^{\ell_1}$) α_1 indexes the diagrams defining the (n_1, \vec{s}_1) -cells for $W_{\ell_1}^{m_1}T_1$, and likewise α_2 indexes the diagrams defining the (n_2, \vec{s}_2) -cells for $W_{\ell_2}^{m_2}T_2$, the top map is the wedge of maps $\phi_{(n_1, \vec{s}_1)} \square \phi_{(n_2, \vec{s}_2)}$ and the lefthand vertical map is induced by the attaching maps for the cells. The associativity isomorphism for the smash product of symmetric spectra gives a canonical isomorphism between $\phi_{(n_1, \vec{s}_1)} \square \phi_{(n_2, \vec{s}_2)}$ and $\phi_{(n_1+n_2, \vec{s}_1 \cdot \vec{s}_2)}$ where “ $\vec{s}_1 \cdot \vec{s}_2$ ” denotes concatenation of sequences. Using this isomorphism on the domain, the inductively defined map $F^{\ell-1} \rightarrow W_{\ell-1}^{m_1+m_2}(T_1 \wedge T_2)$, and the augmentation $F^{\ell} \rightarrow T_1 \wedge T_2$, we get a function from the set of cells in the diagram above to the set of cells constructing $W_{\ell}^{m_1+m_2}(T_1 \wedge T_2)$ from $W_{\ell-1}^{m_1+m_2}(T_1 \wedge T_2)$. We then get an induced map on the pushout $F^{\ell} \rightarrow W_{\ell}^{m_1+m_2}(T_1 \wedge T_2)$. Using the identification of $W^{m_1}T_1 \wedge W^{m_2}T_2$ as the colimit of F^{ℓ} , we get an induced map $W^{m_1}T_1 \wedge W^{m_2}T_2 \rightarrow W^{m_1+m_2}(T_1 \wedge T_2)$.

The following is clear from the construction.

Proposition 8.5. *For all symmetric spectra T_1, T_2 , and integers m_1, m_2 , the following diagram commutes.*

$$\begin{array}{ccccc}
W^{m_1+1}T_1 \wedge W^{m_2+1}T_2 & \longrightarrow & W^{m_1+m_2+2}(T_1 \wedge T_2) & \longleftarrow & W^{m_1+1}T_1 \wedge W^{m_2+1}T_2 \\
\downarrow & & \downarrow & & \downarrow \\
W^{m_1+1}T_1 \wedge W^{m_2}T_2 & \longrightarrow & W^{m_1+m_2+1}(T_1 \wedge T_2) & \longleftarrow & W^{m_1}T_1 \wedge W^{m_2+1}T_2 \\
\downarrow & & \downarrow & & \downarrow \\
W^{m_1}T_1 \wedge W^{m_2}T_2 & \longrightarrow & W^{m_1+m_2}(T_1 \wedge T_2) & \longleftarrow & W^{m_1}T_1 \wedge W^{m_2}T_2 \\
& \searrow & \downarrow & \swarrow & \\
& & T_1 \wedge T_2 & &
\end{array}$$

Remark 8.6. By construction, each $W^m T$ is a cell complex for the cells described above and each map $W^m T \rightarrow W^{m-1} T$ is the inclusion of a subcomplex. If we let $W^{-\infty} T = \text{colim } W^{-m} T$, then $W^{-\infty} T$ is itself a cell complex, and in particular a cofibrant symmetric spectrum. The smash product and restriction diagrams above then simplify to make $W^{-\infty}$ into a monoidal (but not symmetric monoidal) cofibrant approximation functor for symmetric spectra (in both the level and stable model structures).

9. THE TRANSFER IN THOMASON'S DESCENT SPECTRAL SEQUENCE

In this section, we state and prove the folklore result establishing the compatibility of Thomason's descent spectral sequence with the transfer; we used this in Section 1 to construct the canonical map $\text{Fib}(\kappa) \rightarrow \Sigma^{-1} I_{\mathbb{Z}_p} \mathbb{S}$ in the duality pairing.

We consider the case of a finite étale map $f: Y \rightarrow X$. In that case an algebraic vector bundle over Y by restriction becomes a vector bundle over X ; this induces the K -theory transfer. For a sheaf of abelian groups \mathcal{F} on $X_{\text{ét}}$, we have a canonical map [2, IX.(5.1.2)]

$$tr: f_* f^* \mathcal{F} \longrightarrow \mathcal{F}$$

characterized by performing the codiagonal map on stalks

$$(f_* f^* \mathcal{F})_{\bar{x}} \xrightarrow{\cong} \bigoplus_{\bar{y} \in f^{-1}(\bar{x})} (f_* \mathcal{F})_{\bar{y}} \xrightarrow{\cong} \bigoplus_{\#(f^{-1}(\bar{x}))} \mathcal{F}_{\bar{x}} \longrightarrow \mathcal{F}_{\bar{x}}$$

under the canonical identifications (where $y \in f^{-1}(\bar{x})$ is shorthand for $\bar{y}: k(x)^s \rightarrow Y$ is a map whose composite with f is \bar{x}). The transfer in étale cohomology is the composite map

$$H_{\text{ét}}^*(Y; f^* \mathcal{F}) \cong H_{\text{ét}}^*(X; Rf_*(f^* \mathcal{F})) \cong H_{\text{ét}}^*(X; f_* f^* \mathcal{F}) \xrightarrow{tr_*} H_{\text{ét}}^*(X; \mathcal{F}).$$

In the case when X and Y are fields, this becomes the usual group cohomology transfer under the canonical identification of étale cohomology with Galois cohomology. We prove the following theorem.

Theorem 9.1. *Let $Y \rightarrow X$ be a finite étale map of schemes satisfying the hypotheses of [43, 4.1]. Then the étale cohomology transfer on E_2 -terms induces a map of descent spectral sequences that converges to the $K(1)$ -local K -theory transfer.*

The proof occupies the remainder of the section.

First consider the presheaves of symmetric spectra $\mathcal{K}_{X_{\text{ét}}}^{/p^n}$ on $X_{\text{ét}}$ and $\mathcal{K}_{Y_{\text{ét}}}^{/p^n}$ on $Y_{\text{ét}}$. Functoriality of $\mathcal{K}^{/p^n}$ for all maps induces a map

$$\phi: f^*(\mathcal{K}_{X_{\text{ét}}}^{/p^n}) \longrightarrow \mathcal{K}_{Y_{\text{ét}}}^{/p^n}.$$

of presheaves of symmetric spectra on $Y_{\text{ét}}$. Since under the hypotheses on X and Y , ϕ is a weak equivalence on stalks and $\mathcal{K}_{Y_{\text{ét}}}^{/p^n}$ satisfies hypercover descent, $f_*\mathcal{K}_{Y_{\text{ét}}}^{/p^n}$ gives a model for the derived functor $Rf_*(f^*\mathcal{K}_{X_{\text{ét}}}^{/p^n})$. This model does not admit a rigid version of the K -theory transfer, but we can produce a model that does. We prove the following lemma at the end of the section.

Lemma 9.2. *There exists an objectwise fibrant presheaf of symmetric spectra $\mathcal{K}_f^{/p^n}$ on $X_{\text{ét}}$ with an objectwise levelwise equivalence of presheaves of symmetric spectra $\mathcal{K}_f^{/p^n} \rightarrow f_*\mathcal{K}_{Y_{\text{ét}}}^{/p^n}$ on $X_{\text{ét}}$ and a map of presheaves of symmetric spectra*

$$tr: \mathcal{K}_f^{/p^n} \longrightarrow \mathcal{K}_{X_{\text{ét}}}^{/p^n}$$

on $X_{\text{ét}}$ such that the composite map in the stable category

$$\mathcal{K}^{/p^n}(Y \times_X V) \simeq \mathcal{K}_f^{/p^n}(V) \longrightarrow \mathcal{K}^{/p^n}(V)$$

is the $K(1)$ -local K -theory transfer for every $V \rightarrow X$ in $X_{\text{ét}}$.

Thomason's descent spectral sequence for Y is the cosimplicial homotopy group spectral sequence for the Godement resolution of $\mathcal{K}_{Y_{\text{ét}}}^{/p^n}$. If we choose geometric points for Y consistent with those for X (meaning that $f(\bar{y}) = \overline{f(y)}$), then we have a commutative diagram

$$\begin{array}{ccc} \bar{Y}_{\text{disc}} & \xrightarrow{q} & Y \\ g \downarrow & & \downarrow f \\ \bar{X}_{\text{disc}} & \xrightarrow{p} & X \end{array}$$

where \bar{X}_{disc} and \bar{Y}_{disc} denote disjoint union of the chosen geometric points as in Section 5.3. Define

$$g_*: \mathcal{S}^{\bar{Y}_{\text{disc}}} \longrightarrow \mathcal{S}^{\bar{X}_{\text{disc}}}$$

to be the functor such that the \bar{x} coordinate of $g_*((T_{\bar{y}}))$ is the product of $T_{\bar{y}}$ over $g(\bar{y}) = \bar{x}$. By inspection, the modified direct and inverse image functors from Section 5.3

$$(9.3) \quad p^*: \mathcal{P}(X_{\text{ét}}, \mathcal{S}) \rightleftarrows \mathcal{S}^{\bar{X}_{\text{disc}}} : p_* \quad q^*: \mathcal{P}(Y_{\text{ét}}, \mathcal{S}) \rightleftarrows \mathcal{S}^{\bar{Y}_{\text{disc}}} : q_*$$

admit a base change morphism

$$\chi: p^*f_* \longrightarrow g_*q^*$$

and a functoriality isomorphism

$$\theta: p_*g_* \xrightarrow{\cong} f_*q_*.$$

Moreover, χ and θ are compatible with the adjunctions (9.3) in the sense that

- (i) for any presheaf of symmetric spectra \mathcal{F} on $Y_{\text{ét}}$, the composite map

$$f_*\mathcal{F} \xrightarrow{\eta_X} p_*p^*f_*\mathcal{F} \xrightarrow{p_*\chi} p_*g_*q^*\mathcal{F} \xrightarrow{\theta} f_*q_*q^*\mathcal{F}$$

is $f_*\eta_Y$ where η_X denotes the unit of the p^*, p_* adjunction and η_Y denotes the unit of the q^*, q_* adjunction, and

- (ii) for any object \mathcal{F} of $\mathcal{S}^{\bar{Y}_{\text{disc}}}$, the composite map

$$p^*p_*g_*\mathcal{F} \xrightarrow{p^*\theta} p^*f_*q_*\mathcal{F} \xrightarrow{\chi} g_*q^*q_*\mathcal{F} \xrightarrow{g_*\epsilon_Y} g_*\mathcal{F}$$

is ϵ_X , where ϵ_X denotes the counit of the p^*, p_* adjunction and ϵ_Y denotes the counit of the q^*, q_* adjunction.

We then have the following immediate consequence.

Proposition 9.4. *The natural transformations χ and θ induce a natural transformation of cosimplicial presheaves of symmetric spectra on $X_{\text{ét}}$*

$$G_{X_{\text{ét}}}^\bullet f_* \longrightarrow f_* G_{Y_{\text{ét}}}^\bullet.$$

Returning to the case of $\mathcal{K}_{Y_{\text{ét}}}^{/p^n}$, the cosimplicial filtrations on $G_{X_{\text{ét}}}(f_*\mathcal{K}_{Y_{\text{ét}}}^{/p^n})(X)$ and $G_{Y_{\text{ét}}}(\mathcal{K}_{Y_{\text{ét}}}^{/p^n})(Y) = f_*G_{Y_{\text{ét}}}(\mathcal{K}_{Y_{\text{ét}}}^{/p^n})(X)$ induce homotopy group spectral sequences and the maps above induce maps on spectral sequences. Note that for varying s and fixed t , the $E_1^{s,t}$ -term with E_1 differential for each spectral sequence is the global sections of a flasque resolution

$$\begin{aligned} I^s &= N^s(\tilde{\pi}_t(G_{X_{\text{ét}}}^\bullet(f_*\mathcal{K}_{Y_{\text{ét}}}^{/p^n}))), \quad \text{or} \\ I^s &= N^s(f_*\tilde{\pi}_t(G_{Y_{\text{ét}}}^\bullet(\mathcal{K}_{Y_{\text{ét}}}^{/p^n}))) \end{aligned}$$

of $\tilde{\pi}_t(f_*\mathcal{K}_{Y_{\text{ét}}}^{/p^n})$ (where N^s denotes the intersection of the kernel of the degeneracies in a cosimplicial object). It follows that the map of spectral sequences is an isomorphism on E_2 . Thus, the map in Proposition 9.4 induces an isomorphism from E_2 onward between the homotopy group spectral sequence for the cosimplicial symmetric spectrum $G_{X_{\text{ét}}}^\bullet(\mathcal{K}_f^{/p^n})(X)$ and Thomason's descent spectral sequence for the mod p^n $K(1)$ -local K -theory of Y .

The objectwise levelwise weak equivalence $\mathcal{K}_f^{/p^n} \rightarrow f_*\mathcal{K}_{Y_{\text{ét}}}^{/p^n}$ induces an isomorphism on E_1 -terms for the homotopy spectral sequence. Combining this isomorphism with the isomorphism of the previous paragraph, the map of presheaves of symmetric spectra on $X_{\text{ét}}$ from $\mathcal{K}_f^{/p^n}$ to $\mathcal{K}_{X_{\text{ét}}}^{/p^n}$ induces a map of spectral sequences (from E_2 onward) from Thomason's descent spectral sequence for $\mathcal{K}^{/p^n}(Y)$ to the one for $\mathcal{K}^{/p^n}(X)$, converging to the $K(1)$ -local K -theory transfer. By inspection, the map on E_2 -terms is the étale cohomology transfer.

This completes the proof of Theorem 9.1 except Lemma 9.2, which we now prove.

Proof of Lemma 9.2. For $V \rightarrow X$ an object in $X_{\text{ét}}$, let $\mathcal{C}_f(V)$ be the category where an object consists of a big vector bundle η on $V \times_X Y$, a big vector bundle γ on V and an isomorphism $\gamma_V \cong f_*\eta_{(V \times_X Y)}$, where γ_V and $\eta_{(V \times_X Y)}$ denote the restriction to (small) vector bundles on V and $V \times_X Y$, respectively, and $f_*\eta_{(V \times_X Y)}$ is the pushdown (restriction of scalars) of $\eta_{(V \times_X Y)}$ to a (small) vector bundle on V . We have a forgetful functor u from $\mathcal{C}_f(V)$ to the category $\mathcal{V}_{\text{Big}}(V \times_X Y)$ of big vector bundles on $V \times_X Y$ and a “transfer” functor $tr: \mathcal{C}_f(V) \rightarrow \mathcal{V}_{\text{Big}}(V)$, defined by remembering just η or just γ , respectively. We make $\mathcal{C}_f(V)$ a Waldhausen category

by defining the weak equivalences and cofibrations via the forgetful functor u . We can then form the Waldhausen K -theory of $\mathcal{C}_f(V)$ as in Section 5.5. The functors u and tr are exact, and we get induced maps on K -theory spectra. We produce $\mathcal{K}_f^{/p^n}$ by the same recipe as $\mathcal{K}^{/p^n}$ as in Section 5.5 except we work in the vanilla category of symmetric spectra as the K -theory of $\mathcal{C}_f(V)$ does not have a canonical A_2 -structure (and the transfer map of spectral sequences is not multiplicative even on E_2 -terms). There is a canonical map from functorial factorization in symmetric spectra to the functorial factorization in A_2 symmetric spectra. This constructs an objectwise weak equivalence $u: \mathcal{K}_f^{/p^n} \rightarrow f_*\mathcal{K}_{Y_{\text{ét}}}^{/p^n}$ and a map $tr: \mathcal{K}_f^{/p^n} \rightarrow \mathcal{K}_{X_{\text{ét}}}^{/p^n}$ inducing the K -theory transfer. \square

REFERENCES

- [1] D. W. Anderson. There are no phantom cohomology operations in K -theory. *Pacific J. Math.*, 107(2):279–306, 1983.
- [2] M. Artin, A. Grothendieck, and J.-L. Verdier. *Théorie des topos and cohomologie étale des schémas*, volume 269, 270, 305 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1972–1973.
- [3] M. Artin and B. Mazur. *Etale homotopy*, volume 100 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986. Reprint of the 1969 original.
- [4] Andrew J. Blumberg and Michael A. Mandell. Derived Koszul duality and involutions in the algebraic K -theory of spaces. *J. Topol.*, 4(2):327–342, 2011.
- [5] Andrew J. Blumberg and Michael A. Mandell. Localization theorems in topological Hochschild homology and topological cyclic homology. *Geom. Topol.*, 16(2):1053–1120, 2012.
- [6] Andrew J. Blumberg and Michael A. Mandell. The homotopy groups of the algebraic K -theory of the sphere spectrum. Preprint, arXiv:1408.0133, 2014.
- [7] M. Bökstedt and I. Madsen. Topological cyclic homology of the integers. *Astérisque*, (226):57–143, 1994. K -theory (Strasbourg, 1992).
- [8] A. K. Bousfield. The localization of spectra with respect to homology. *Topology*, 18(4):257–281, 1979.
- [9] A. K. Bousfield and D. M. Kan. A second quadrant homotopy spectral sequence. *Trans. Amer. Math. Soc.*, 177:305–318, 1973.
- [10] C. Breuil. The emerging p -adic Langlands program. In *Proceedings of the International Congress of Mathematicians. Volume II*, pages 203–230. Hindustan Book Agency, New Delhi, 2010.
- [11] Frank Calegari. The stable homology of congruence subgroups. *Geom. Topol.*, 19(6):3149–3191, 2015.
- [12] Frank Calegari and Matthew Emerton. Completed cohomology—a survey. In *Non-abelian fundamental groups and Iwasawa theory*, volume 393 of *London Math. Soc. Lecture Note Ser.*, pages 239–257. Cambridge Univ. Press, Cambridge, 2012.
- [13] Frank Calegari and Matthew Emerton. Homological stability for completed homology. *Math. Ann.*, 364(3-4):1025–1041, 2016.
- [14] D. Clausen. Arithmetic duality in algebraic K -theory. MIT thesis, 2013. See also <http://dtclausen.tumblr.com/errata>.
- [15] Ethan S. Devinatz and Michael J. Hopkins. Homotopy fixed point spectra for closed subgroups of the Morava stabilizer groups. *Topology*, 43(1):1–47, 2004.
- [16] D. Dugger. Multiplicative structures on homotopy spectral sequences, part I. Preprint, arXiv:math/0305173, 2003.
- [17] D. Dugger. Multiplicative structures on homotopy spectral sequences, part II. Preprint, arXiv:math/0305187, 2003.
- [18] Daniel Dugger, Sharon Hollander, and Daniel C. Isaksen. Hypercovers and simplicial presheaves. *Math. Proc. Cambridge Philos. Soc.*, 136(1):9–51, 2004.
- [19] W. G. Dwyer and S. A. Mitchell. On the K -theory spectrum of a ring of algebraic integers. *K-Theory*, 14(3):201–263, 1998.
- [20] A. D. Elmendorf and M. A. Mandell. Rings, modules, and algebras in infinite loop space theory. *Adv. Math.*, 205(1):163–228, 2006.

- [21] M. Emerton. Completed cohomology and the p -adic Langlands program. In *Proceedings of the International Congress of Mathematicians. Volume II*, pages 319–342. Kyung Moon SA Co. Ltd., Seoul, S. Korea, 2014.
- [22] Eric M. Friedlander and Andrei Suslin. The spectral sequence relating algebraic K -theory to motivic cohomology. *Ann. Sci. École Norm. Sup. (4)*, 35(6):773–875, 2002.
- [23] Daniel R. Grayson. Weight filtrations via commuting automorphisms. *K-Theory*, 9(2):139–172, 1995.
- [24] J. P. C. Greenlees and J. P. May. Generalized Tate cohomology. *Mem. Amer. Math. Soc.*, 113(543):viii+178, 1995.
- [25] Lars Hesselholt and Ib Madsen. Cyclic polytopes and the K -theory of truncated polynomial algebras. *Invent. Math.*, 130(1):73–97, 1997.
- [26] Lars Hesselholt and Ib Madsen. On the K -theory of finite algebras over Witt vectors of perfect fields. *Topology*, 36(1):29–101, 1997.
- [27] Lars Hesselholt and Ib Madsen. On the K -theory of local fields. *Ann. of Math. (2)*, 158(1):1–113, 2003.
- [28] Kenkichi Iwasawa. On p -adic L -functions. *Ann. of Math. (2)*, 89:198–205, 1969.
- [29] J. F. Jardine. Simplicial presheaves. *J. Pure Appl. Algebra*, 47(1):35–87, 1987.
- [30] J. F. Jardine. Stable homotopy theory of simplicial presheaves. *Canad. J. Math.*, 39(3):733–747, 1987.
- [31] J. F. Jardine. Presheaves of symmetric spectra. *J. Pure Appl. Algebra*, 150(2):137–154, 2000.
- [32] J. F. Jardine. *Generalized étale cohomology theories*. Modern Birkhäuser Classics. Birkhäuser/Springer Basel AG, Basel, 2010. Reprint of the 1997 edition [MR1437604].
- [33] Karlheinz Knapp. Anderson duality in K -theory and $\mathrm{Im}(J)$ -theory. *K-Theory*, 18(2):137–159, 1999.
- [34] J. S. Milne. *Arithmetic duality theorems*. BookSurge, LLC, Charleston, SC, second edition, 2006.
- [35] James S. Milne. *Étale cohomology*, volume 33 of *Princeton Mathematical Series*. Princeton University Press, Princeton, N.J., 1980.
- [36] Stephen A. Mitchell. On p -adic topological K -theory. In *Algebraic K-theory and algebraic topology (Lake Louise, AB, 1991)*, volume 407 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 197–204. Kluwer Acad. Publ., Dordrecht, 1993.
- [37] Jürgen Neukirch, Alexander Schmidt, and Kay Wingberg. *Cohomology of number fields*, volume 323 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 2008.
- [38] John Rognes. Two-primary algebraic K -theory of pointed spaces. *Topology*, 41(5):873–926, 2002.
- [39] John Rognes. The smooth Whitehead spectrum of a point at odd regular primes. *Geom. Topol.*, 7:155–184 (electronic), 2003.
- [40] T. Schlank and V. Stojanoska. Arithmetic duality for spectra. In progress, 2015.
- [41] J.-P. Serre. Local class field theory. In *Algebraic Number Theory (Proc. Instructional Conf., Brighton, 1965)*, pages 128–161. Thompson, Washington, D.C., 1967.
- [42] J.P. Serre. *Local fields*, volume 67 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1979.
- [43] R. W. Thomason. Algebraic K -theory and étale cohomology. *Ann. Sci. École Norm. Sup. (4)*, 18(3):437–552, 1985.
- [44] F. Waldhausen, B. Jähren, and J. Rognes. *Spaces of PL manifolds and categories of simple maps*, volume 186 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, 2013.
- [45] Charles A. Weibel. *The K-book*, volume 145 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2013. An introduction to algebraic K -theory.

DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF TEXAS, AUSTIN, TX 78712
E-mail address: blumberg@math.utexas.edu

DEPARTMENT OF MATHEMATICS, INDIANA UNIVERSITY, BLOOMINGTON, IN 47405
E-mail address: mmandell@indiana.edu